Regularization parameters for the self-force of a scalar particle in a general orbit about a Schwarzschild black hole

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The interaction of a charged particle with its own field results in the "self-force" on the particle, which includes but is more general than the radiation reaction force. In the vicinity of the particle in curved spacetime, one may follow Dirac and split the retarded field of the particle into two parts, (1) the singular source field, $\sim q/r$, and (2) the regular remainder field. The singular source field exerts no force on the particle, and the self-force is entirely caused by the regular remainder. We describe an elementary multipole decomposition of the singular source field which allows for the calculation of the self-force on a scalar-charged particle orbiting a Schwarzschild black hole.

I. INTRODUCTION

According to the equivalence principle in general relativity, a particle of infinitesimal mass orbits a black hole of large mass along a geodesic worldline Γ in the background spacetime determined by the large mass alone. For a particle of small but finite mass, the orbit is no longer a geodesic in the background of the large mass because the particle perturbs the spacetime geometry. This perturbation due to the presence of the smaller mass modifies the orbit of the particle from an original geodesic in the background. The difference of the actual orbit from a geodesic in the background is said to result from the interaction of the moving particle with its own gravitational field, which is called a *self-force* **(author?)** [1].

Historically, Dirac **(author?)** [2] first gave the analysis of the self-force for the electromagnetic field of a particle in flat spacetime. He was able to approach the problem in a perturbative scheme by allowing the particle's size to remain finite and invoking the conservation of the stress-energy tensor inside a narrow world tube surrounding the particle's worldline. Dewitt and Brehme **(author?)** [3] extended Dirac's problem to curved spacetime. Mino, Sasaki, and Tanaka **(author?)** [4] generalized it for the gravitational field self-force. Quinn and Wald **(author?)** [5] and Quinn **(author?)** [6] worked out similar schemes for the gravitational, electromagnetic, and scalar field self-forces by taking an axiomatic approach.

In Dirac's **(author?)** [2] flat spacetime problem, the retarded field is decomposed into two parts: (i) The first part is the "mean of the advanced and retarded fields" which is a solution of the inhomogeneous field equation resembling the Coulomb q/r piece of the scalar potential near the particle, (ii) The second part is a "radiation" field which is a homogeneous solution of Maxwell's equations. Dirac describes the self-force as the interaction of the particle with the radiation field, a well-defined vacuum field solution.

In the analyses of the self-force in curved spacetime, the Hadamard form of Green's function **(author?)** [3] is employed to describe the retarded field of the particle. Traditionally, taking the scalar field case for example, the retarded Green's function $G^{\rm ret}(p,p')$ is divided into "direct" and "tail" parts: (i) The first part has support only on the past null cone of the field point p, (ii) The second part has the support inside the past null cone due to the presence of the curvature of spacetime. Accordingly, the self-force on the particle would consist of two pieces: (i) The first piece comes from the direct part of the field and the acceleration of the worldline in the background geometry; this corresponds to Abraham-Lorentz-Dirac (ALD) force in flat spacetime, (ii) The second piece comes from the tail part of the field and is present in curved spacetime. Thus, the description of the self-force in curved spacetime should reduce to Dirac's result in the flat spacetime limit. In this approach, the self-force is considered to result via

$$\mathcal{F}_a = q\nabla_a\psi, \tag{1}$$

from the interaction of the particle with the quantity **(author?)** [1]

$$\psi^{\rm self} = \psi^{\rm ret} - \psi^{\rm dir}. \tag{2}$$

Although this traditional approach provides adequate methods to compute the self-force, it does not share the physical simplicity of Dirac's analysis where the force is described entirely in terms of an identifiable, vacuum solution of the field equations: unlike Dirac's radiation field, the quantity in Eq. (**??**) is not a homogeneous solution of the field equation $\nabla^2\psi = -4\pi\varrho$. Moreover, the integral term in this quantity comes from the tail part of the Green's function and is generally not differentiable on the worldline if the Ricci scalar of the background is not zero (similarly, the electromagnetic potential $A_a^{\rm tail}$ and the gravitational metric perturbation $h_{ab}^{\rm tail}$ are not differentiable at the point

of the particle unless $\left(R_{ab} - \frac{1}{6}g_{ab}R\right)u^b$ and $R_{cadb}u^cu^d$, respectively are zero in the background **(author?)** [7]). Thus, some version of averaging process must be invoked to make sense of the self-force.

In this paper an alternative method to split the retarded field ψ^{ret} as suggested by Ref. **(author?)** [1] is used such that the splits resemble those by Dirac: (i) The first part, the *Singular Source field* ψ^{S}, is an inhomogeneous field similar to the Coulomb potential piece and exerts no force on the particle, (ii) The second part, the *Regular Remainder field* ψ^{R}, being an homogeneous solution of the field equation, is analogous to Dirac's radiation field and entirely responsible for the self-force. This alternative method is reviewed briefly in Section II.

In Section III we give a brief overview of the mode-sum decomposition scheme to evaluate the self-force. We consider a particle with a scalar charge q in general geodesic motion about a Schwarzschild black hole in this paper. A spherical harmonic decomposition of both ψ^{ret} and ψ^{S} having this condition should be performed to provide the multipole components of each. Then, the mode by mode sum of the difference of these components determines ψ^{R}, and, thence the self-force. The multipole components of ψ^{ret} can be determined numerically while the multipole components of ψ^{S} are derived analytically. In particular, the multipole moments of ψ^{S} are generically referred to as the *Regularization Parameters*; the entire paper focuses on the analytical task to find these regularization parameters. We summarize our analytical results at the end of the Section.

The description of ψ^{S} becomes advantageously simple in a specially chosen co-moving frame: the THZ normal coordinates of this frame, introduced by Thorne and Hartle **(author?)** [8] and extended by Zhang **(author?)** [9] are locally inertial on a geodesic. In Section IV we obtain a simple form of ψ^{S} using the THZ coordinates and then re-express it in terms of the background, i.e. Schwarzschild coordinates via the coordinate transformation between the THZ and the background coordinates.

Section V outlines the derivation of the regularization parameters given below in Eqs. (11) to (18). These results are in agreement with Barack and Ori **(author?)** [10] and Mino, Nakano, and Sasaki **(author?)** [11].

In Appendix A we give a detailed description of the THZ coordinates. Appendix B provides some mathematical details concerning the hypergeometric functions and the different representations of the regularization parameters in connection with them.

Notation: $(t,\, r,\, \theta,\, \phi)$ are the usual Schwarzschild coordinates. $(T,\, X,\, Y,\, Z)$ are the initial static normal coordinates, reshaped out of the Schwarzschild coordinates. $(\mathcal{T},\, \mathcal{X},\, \mathcal{Y},\, \mathcal{Z})$ are the THZ normal coordinates attached to the geodesic Γ, and $\rho \equiv \sqrt{\mathcal{X}^2 + \mathcal{Y}^2 + \mathcal{Z}^2}$. The points p and p' refer to a field point and a source point on the worldline of the particle, respectively. In the coincidence limit $p \to p'$.

II. DECOMPOSITION OF THE RETARDED FIELD

The recent analysis of the Green's function decomposition by Detweiler and Whiting **(author?)** [1] shows an alternative way to split the retarded field into two parts

$$\psi^{\mathrm{ret}} = \psi^{\mathrm{S}} + \psi^{\mathrm{R}}, \tag{3}$$

where ψ^{S} and ψ^{R} are named the **S**ingular **S**ource field and the **R**egular **R**emainder field, respectively. The source function for a point particle on the worldline Γ is $\varrho(p) = q\int(-g)^{-1/2}\delta^4(p - p'(\tau'))d\tau'$. Like ψ^{ret}, ψ^{S} is an inhomogeneous solution of the scalar field equation

$$\nabla^2\psi = -4\pi\varrho \tag{4}$$

in the neighborhood of the particle. And ψ^{S} is determined in the neighborhood of the particle's worldline entirely by local analysis. ψ^{R}, defined by Eq. (3), is then necessarily a homogeneous solution and is therefore expected to be differentiable on Γ. According to Ref. **(author?)** [1], ψ^{R} will formally give the correct self-force when substituted on the right hand side of Eq. (1) in place of ψ^{tail}. In this paper we adopt this decomposition, and try to determine an analytical approximation to ψ^{S}, which is to be subtracted from ψ^{ret} for an explicit computation of the self-force.

III. MODE-SUM DECOMPOSITION AND REGULARIZATION PARAMETERS

By Eq. (1) the self-force can be formally evaluated from

$$\mathcal{F}_a^{\mathrm{self}} \;=\; \lim_{p\to p'}\left[\mathcal{F}_a^{\mathrm{ret}}(p) - \mathcal{F}_a^{\mathrm{S}}(p)\right] = \lim_{p\to p'}\mathcal{F}_a^{\mathrm{R}}(p)$$

$$\equiv\ q \lim_{p\to p'} \nabla_a \psi^{\mathrm{R}} = q \lim_{p\to p'} \nabla_a (\psi^{\mathrm{ret}} - \psi^{\mathrm{S}}), \tag{5}$$

where p' is the event on Γ where the self-force is to be determined and p is an event in the neighborhood of p'. For use of this equation, both $\mathcal{F}^{\mathrm{ret}}_a(p)$ and $\mathcal{F}^{\mathrm{S}}_a(p)$ would be expanded into multipole ℓ-modes, with $\mathcal{F}^{\mathrm{ret}}_{\ell a}(p)$ determined numerically.

Typically, if the background geometry is Schwarzschild spacetime, the source function $\varrho(p)$ is expanded in terms of spherical harmonics, and then similar expansion for ψ^{ret} is made

$$\psi^{\mathrm{ret}} = \sum_{\ell m} \psi^{\mathrm{ret}}_{\ell m}(r,t) Y_{\ell m}(\theta, \phi), \tag{6}$$

where $\psi^{\mathrm{ret}}_{\ell m}(r,t)$ is found numerically. The individual ℓm components of ψ^{ret} in this expansion are finite at the location of the particle even though their sum is singular. Then, $\mathcal{F}^{\mathrm{ret}}_{\ell a}$ is finite and can be obtained as

$$\mathcal{F}^{\mathrm{ret}}_{\ell a} = q \nabla_a \sum_m \psi^{\mathrm{ret}}_{\ell m} Y_{\ell m}, \tag{7}$$

where a represents each component of t, r, ϕ, θ in the Schwarzschild geometry.

The singular source field ψ^{S} is determined analytically in the neighborhood of the particle's worldline via local analysis (see Section IV). Then, $\nabla_a \psi^{\mathrm{S}}$ is evaluated and the mode-sum decomposition of this quantity is performed to provide

$$\mathcal{F}^{\mathrm{S}}_{\ell a} = q \nabla_a \sum_m \psi^{\mathrm{S}}_{\ell m} Y_{\ell m}, \tag{8}$$

which is also finite at the location of the particle.

Then, using Eqs. (5), (7), and (8) the self-force is finally

$$\begin{aligned} \mathcal{F}^{\mathrm{self}}_a &= \sum_\ell \lim_{p\to p'} \left[\mathcal{F}^{\mathrm{ret}}_{\ell a}(p) - \mathcal{F}^{\mathrm{S}}_{\ell a}(p)\right] \\ &= q \sum_\ell \lim_{p\to p'} \nabla_a \sum_m (\psi^{\mathrm{ret}}_{\ell m} - \psi^{\mathrm{S}}_{\ell m}) Y_{\ell m} \end{aligned} \tag{9}$$

evaluated at the location of the particle.

In Section V the regularization parameters are derived from the multipole components of $\nabla_a \psi^{\mathrm{S}}$ evaluated at the source point and are used to control both singular behavior and differentiability. We follow Barack and Ori **(author?)** [12] in defining the regularization counter terms, except that the singular source field ψ^{S} is used in place of ψ^{dir}

$$\lim_{p\to p'} \mathcal{F}^{\mathrm{S}}_{\ell a} = \left(\ell + \frac{1}{2}\right) A_a + B_a + \frac{C_a}{\ell + \frac{1}{2}} + O(\ell^{-2}), \tag{10}$$

and show

$$A_t = \mathrm{sgn}(\Delta) \frac{q^2}{r_{\mathrm{o}}^2} \frac{\dot{r}}{1 + J^2/r_{\mathrm{o}}^2}, \tag{11}$$

$$A_r = -\mathrm{sgn}(\Delta) \frac{q^2}{r_{\mathrm{o}}^2} \frac{E\left(1 - \frac{2M}{r_{\mathrm{o}}}\right)^{-1}}{1 + J^2/r_{\mathrm{o}}^2}, \tag{12}$$

$$A_\phi = 0, \tag{13}$$

$$B_t = \frac{q^2}{r_{\mathrm{o}}^2} E \dot{r} \left[\frac{F_{3/2}}{(1 + J^2/r_{\mathrm{o}}^2)^{3/2}} - \frac{3F_{5/2}}{2\,(1 + J^2/r_{\mathrm{o}}^2)^{5/2}}\right], \tag{14}$$

$$B_r = \frac{q^2}{r_o^2}\left\{-\frac{F_{1/2}}{(1+J^2/r_o^2)^{1/2}} + \frac{[1-2\left(1-\frac{2M}{r_o}\right)^{-1}\dot{r}^2]F_{3/2}}{2\,(1+J^2/r_o^2)^{3/2}} + \frac{3\left(1-\frac{2M}{r_o}\right)^{-1}\dot{r}^2 F_{5/2}}{2\,(1+J^2/r_o^2)^{5/2}}\right\}, \tag{15}$$

$$B_\phi = \frac{q^2}{J}\dot{r}\left[\frac{F_{1/2}-F_{3/2}}{(1+J^2/r_o^2)^{1/2}} + \frac{3(F_{5/2}-F_{3/2})}{2\,(1+J^2/r_o^2)^{3/2}}\right], \tag{16}$$

$$C_t = C_r = C_\phi = 0, \tag{17}$$

$$A_\theta = B_\theta = C_\theta = 0, \tag{18}$$

where $\Delta \equiv r - r_o$, $E \equiv -u_t = (1-2M/r_o)\,(dt/d\tau)_o$ (τ: proper time) and $J \equiv u_\phi = r_o^2\,(d\phi/d\tau)_o$ are the conserved energy and angular momentum, respectively, and $\dot{r} \equiv u^r = (dr/d\tau)_o$. Also, shorthand notations are used for the hypergeometric functions, $F_p \equiv {}_2F_1\left(p, \frac{1}{2}; 1; \frac{J^2}{r_o^2+J^2}\right)$ (see Appendix B for more details about the hypergeometric functions and the representations of the regularization parameters in connection with them).

IV. DETERMINATION OF $\psi^{\rm S}$ VIA THE THZ NORMAL COORDINATES

It was mentioned earlier in Sections II and III that $\psi^{\rm S}$ is determined in the neighborhood of the particle's worldline entirely by local analysis. When analyzed by some special local coordinate system in which the background geometry looks as flat as possible, the scalar wave equation might take a simple form and $\psi^{\rm S}$ might look like a simple Coulomb potential piece. Detweiler, Messaritaki, and Whiting **(author?)** [7] (cited henceforth as Paper I) show

$$\psi^{\rm S} = q/\rho + O(\rho^2/\mathcal{R}^3), \tag{19}$$

where $\rho = \sqrt{\mathcal{X}^2+\mathcal{Y}^2+\mathcal{Z}^2}$ with $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ being spatial components in that local inertial coordinate system and $\mathcal{R}$ represents a length scale of the background geometry (the smallest of the radius of curvature, the scale of inhomogeneities, and time scale for changes in curvature along Γ).

Describing this special coordinate system more precisely, first, it must be a *normal* coordinate system where on Γ, the metric and its first derivatives match the Minkowski metric, and the coordinate $\mathcal{T}$ measures the proper time. Normal coordinates for a geodesic, however, are not unique, and we use particular ones that were introduced by Thorne and Hartle **(author?)** [8] and extended by Zhang **(author?)** [9] to describe the external multipole moments of a vacuum solution of the Einstein equations: namely, the **THZ NORMAL COORDINATES**.

However, in order to derive the regularization parameters from the multipole components of $\nabla_a\psi^{\rm S}$, ρ in Eq. (19) must be expressed in terms of the coordinates of the original background, which is the Schwarzschild geometry in our problem. Then, this requires us to find the expressions of $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ (THZ) in terms of t, r, ϕ, θ (Schwarzschild): the task here is simply to find the coordinate transformation between two different geometries.

Based on the idea from Weinberg **(author?)** [13], one can achieve this coordinate transformation to the level of accuracy we desire for this particular problem of mode-sum regularization, by taking the following two steps basically:

(i) Find an inertial Cartesian coordinates X^A to redirect the Schwarzschild coordinates x^a by the Taylor's expansion around the location of the particle, x_o^a;

$$X^A = X_o^A + M^A{}_a(x^a - x_o^a) + \frac{1}{2}M^A{}_a\,\Gamma^a_{bc}\big|_o\,(x^b - x_o^b)(x^c - x_o^c) + O[(x-x_o)^3], \tag{20}$$

where we may choose $X_o^A = 0$ and $M^A{}_a = \mathrm{diag}\left[M^T{}_t,\, M^X{}_r,\, M^Y{}_\phi,\, M^Z{}_\theta\right]$ for convenience (this choice will redirect and rescale the Schwarzschild coordinates as $T = M^T{}_t(t-t_o)$, $X = M^X{}_r(r-r_o)$, $Y = M^Y{}_\phi(\phi-\phi_o)$, $Z = M^Z{}_\theta(\theta-\theta_o)$.

(ii) Boost X^A with u^A, the particle's four-velocity at p' as measured in this Cartesian frame, to obtain the final coordinates $\mathcal{X}^{A'}$;

$$\begin{aligned}\mathcal{X}^{A'} &= \Lambda^{A'}{}_A X^A \\ &= \Lambda^{A'}{}_A \left[M^A{}_a(x^a - x^a_{\rm o}) + \frac{1}{2}M^A{}_a\, \Gamma^a_{bc}\big|_{\rm o}\,(x^b - x^b_{\rm o})(x^c - x^c_{\rm o})\right] + O[(x - x_{\rm o})^3],\end{aligned} \tag{21}$$

where

$$\Lambda^{A'}{}_A = \begin{bmatrix} u^T & -u^X & -u^Y & -u^Z \\ & 1 + (u^T - 1)(u^X)^2/u^2 & (u^T - 1)u^X u^Y/u^2 & (u^T - 1)u^X u^Z/u^2 \\ & \text{SYM} & 1 + (u^T - 1)(u^Y)^2/u^2 & (u^T - 1)u^Y u^Z/u^2 \\ & & & 1 + (u^T - 1)(u^Z)^2/u^2 \end{bmatrix} \tag{22}$$

with $u^2 \equiv (u^X)^2 + (u^Y)^2 + (u^Z)^2$ **(author?)** [14].

According to Ref. **(author?)** [13], one can show out of Eq. (21)

$$\begin{aligned}g^{A'B'} &= g^{ab}\frac{\partial \mathcal{X}^{A'}}{\partial x^a}\frac{\partial \mathcal{X}^{B'}}{\partial x^b} \\ &= \eta^{A'B'} + O[(x - x_{\rm o})^2],\ x^a \to x^a_{\rm o},\end{aligned} \tag{23}$$

so that

$$\frac{\partial g^{A'B'}}{\partial \mathcal{X}^{C'}} = O[(x - x_{\rm o})],\ x^a \to x^a_{\rm o}, \tag{24}$$

with the choice of $M^T{}_t = \left(1 - \frac{2M}{r_{\rm o}}\right)^{1/2}$, $M^X{}_r = \left(1 - \frac{2M}{r_{\rm o}}\right)^{-1/2}$, $M^Y{}_\phi = r_{\rm o}\sin\theta_{\rm o}$, $M^Z{}_\theta = -r_{\rm o}$. Eqs. (23) and (24) are the local inertial features as expected for normal coordinates.

To simplify the calculations, we may confine the particle's orbits to the equatorial plane $\theta_{\rm o} = \pi/2$. Then, we have

$$M^A{}_a = \text{diag}\left[f^{1/2},\, f^{-1/2},\, r_{\rm o},\, -r_{\rm o}\right], \tag{25}$$

where $f \equiv \left(1 - \frac{2M}{r_{\rm o}}\right)$. Also, this constraint of the equatorial plane makes $u^Z = 0$. We can rewrite u^A in terms of the Schwarzschild coordinates and the constants of motion,

$$u^A \equiv \left(u^T,\, u^X,\, u^Y,\, u^Z\right) = \left(f^{-1/2}E,\, f^{-1/2}\dot{r},\, \frac{J}{r_{\rm o}},\, 0\right), \tag{26}$$

where $E \equiv -u_t = f\,(dt/d\tau)_{\rm o}$ (τ: proper time) and $J \equiv u_\phi = r^2_{\rm o}\,(d\phi/d\tau)_{\rm o}$ are the conserved energy and angular momentum, respectively, and $\dot{r} \equiv u^r = (dr/d\tau)_{\rm o}$. From this it follows that $u^2 = f^{-1}E^2 - 1$ and we have

$$\Lambda^{A'}{}_A = \begin{bmatrix} f^{-1/2}E & -f^{-1/2}\dot{r} & -J/r_{\rm o} & 0 \\ & 1 + \dot{r}^2/(f^{1/2}E + f) & J\dot{r}/[r_{\rm o}(E + f^{1/2})] & 0 \\ & \text{SYM} & 1 + J^2/[r^2_{\rm o}(f^{-1/2}E + 1)] & 0 \\ & & & 1 \end{bmatrix}. \tag{27}$$

Now we are finally able to express ρ^2 in terms of the Schwarzschild coordinates. Using Eq. (21) one may write

$$\begin{aligned}\rho^2 = \mathcal{X}^I\mathcal{X}_I &= \delta_{IJ}\Lambda^I{}_C\Lambda^J{}_D M^C{}_c M^D{}_d\left[(x^c - x^c_{\rm o})(x^d - x^d_{\rm o}) + \Gamma^c_{ab}\big|_{\rm o}\,(x^a - x^a_{\rm o})(x^b - x^b_{\rm o})(x^d - x^d_{\rm o})\right] \\ &\quad + O[(x - x_{\rm o})^4],\end{aligned} \tag{28}$$

where $I,\, J = 1,\, 2,\, 3$. Then, using Eqs. (25) and (27), Eq. (28) can be eventually expressed as

$$\rho^2 = (E^2 - f)(t - t_{\rm o})^2 - \frac{2E\dot{r}}{f}(t - t_{\rm o})(r - r_{\rm o}) - 2EJ(t - t_{\rm o})(\phi - \phi_{\rm o})$$

$$
\begin{aligned}
&+\frac{1}{f}\left(1+\frac{\dot{r}^2}{f}\right)(r-r_\text{o})^2+\frac{2J\dot{r}}{f}(r-r_\text{o})(\phi-\phi_\text{o})+(r_\text{o}^2+J^2)(\phi-\phi_\text{o})^2+r_\text{o}^2\left(\theta-\frac{\pi}{2}\right)^2\\
&-\frac{ME\dot{r}}{r_\text{o}^2}(t-t_\text{o})^3+\frac{M}{r_\text{o}^2}\left(-1+\frac{2E^2}{f}+\frac{\dot{r}^2}{f}\right)(t-t_\text{o})^2(r-r_\text{o})+\frac{MJ\dot{r}}{r_\text{o}^2}(t-t_\text{o})^2(\phi-\phi_\text{o})\\
&-\frac{ME\dot{r}}{f^2r_\text{o}^2}(t-t_\text{o})(r-r_\text{o})^2-\frac{2(r_\text{o}-M)EJ}{fr_\text{o}^2}(t-t_\text{o})(r-r_\text{o})(\phi-\phi_\text{o})\\
&+r_\text{o}E\dot{r}(t-t_\text{o})(\phi-\phi_\text{o})^2+r_\text{o}E\dot{r}(t-t_\text{o})\left(\theta-\frac{\pi}{2}\right)^2\\
&-\frac{M}{f^2r_\text{o}^2}\left(1+\frac{\dot{r}^2}{f}\right)(r-r_\text{o})^3+\frac{(2r_\text{o}-5M)J\dot{r}}{f^2r_\text{o}^2}(r-r_\text{o})^2(\phi-\phi_\text{o})\\
&+r_\text{o}\left(1-\frac{\dot{r}^2}{f}+\frac{2J^2}{r_\text{o}^2}\right)(r-r_\text{o})(\phi-\phi_\text{o})^2+r_\text{o}\left(1-\frac{\dot{r}^2}{f}\right)(r-r_\text{o})\left(\theta-\frac{\pi}{2}\right)^2\\
&-r_\text{o}J\dot{r}(\phi-\phi_\text{o})^3-r_\text{o}J\dot{r}(\phi-\phi_\text{o})\left(\theta-\frac{\pi}{2}\right)^2+O[(x-x_\text{o})^4]. \qquad (29)
\end{aligned}
$$

Eq. (29) is substituted into Eq. (19) to determine ψ^S in terms of the Schwarzschild coordinates and will serve significantly to derive the regularization parameters in the next section.

In the above analysis we have ignored the term $O[(x-x_\text{o})^3]$ in Eq. (21) and its contribution to ρ^2, which is $O[(x-x_\text{o})^4]$ in Eqs. (28) and (29). To the level of accuracy we desire for the mode-sum regularization in this paper, that is to say, to the determination of C-terms, it is not necessary to specify the term $O[(x-x_\text{o})^4]$ in ρ^2 (hence not necessary to specify $O[(x-x_\text{o})^3]$ in the spatial THZ coordinates $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$). Even without specifying the term $O[(x-x_\text{o})^4]$ in ρ^2, one can prove that C_a-terms in Eq. (10) always vanish (see Subsection V C). In fact, the clarification of $O[(x-x_\text{o})^3]$ for the THZ coordinates in Eq. (21) requires more involved analyses of coordinate transformations, which would be beyond the scope of this paper. Readers may refer to Appendix A for more detailed description of the THZ coordinates, specified up to the quartic order.

V. REGULARIZATION PARAMETERS FOR A GENERAL ORBIT OF THE SCHWARZSCHILD GEOMETRY

In Section IV, we have seen that an approximation to ψ^S is

$$\psi^\text{S}=q/\rho+O(\rho^2/\mathcal{R}^3). \qquad (30)$$

Following Paper I **(author?)** [7], the regularization parameters can be determined from evaluating the multipole components of $\partial_a(q/\rho)$ ($a=t$, r, θ, ϕ for the Schwarzschild background) at the location of the source. The error, $O(\rho^2/\mathcal{R}^3)$ in the above approximation is disregarded since it gives no contribution to $\nabla_a\psi^\text{S}$ as we take the "coincidence limit", $x\to x_\text{o}$, where x denotes a point in the vicinity of the particle and x_o the location of the particle in the Schwarzschild geometry.

In evaluating the multipole components of $\partial_a(q/\rho)$, singularities are expected with certain terms. To help identify those singularities, we introduce an order parameter ϵ which is to be set to unity at the end of a calculation: we attach ϵ^n to each $O[(x-x_\text{o})^n]$ part of ρ^2 in Eq. (29) and may re-express ρ^2 as

$$\rho^2=\epsilon^2\mathcal{P}_\text{II}+\epsilon^3\mathcal{P}_\text{III}+\epsilon^4\mathcal{P}_\text{IV}+O(\epsilon^5), \qquad (31)$$

where $\mathcal{P}_\text{II}$, $\mathcal{P}_\text{III}$, and $\mathcal{P}_\text{IV}$ represent the quadratic, cubic, and quartic order parts of ρ^2, respectively. Here we pretend that the quartic part $\mathcal{P}_\text{IV}$ is also specified: this will help us to perform the structure analysis for C_a-terms later in Subsection V C when we prove that these regularization parameters always vanish.

When we express $\partial_a\,(1/\rho)$ in the Laurent series expansion to identify the terms according to their singular patterns, every denominator of this expansion should take the form of $\mathcal{P}_\text{II}^{n/2}$ ($n=3$, 5, 7). Due to this special position, which becomes singular in the coincidence limit, $\mathcal{P}_\text{II}$ would play a significant role in inducing the multipole decomposition.

However, the quadratic part $\mathcal{P}_\text{II}$, directly taken from Eq. (29), may not be fully ready for this task yet. First, $\phi-\phi_\text{o}$ needs to be decoupled from $r-r_\text{o}$ so that we have independent complete square forms of each, which is necessary for inducing the Legendre polynomial expansions later. Coupling between $t-t_\text{o}$ and $\phi-\phi_\text{o}$ is not significant because upon fixing $t=t_\text{o}$ all terms having $t-t_\text{o}$ will vanish. Thus, we reshape our quadratic term in Eq. (29) into

$$\mathcal{P}_\text{II} \;=\; (E^2-f)(t-t_\text{o})^2-\frac{2E\dot{r}r_\text{o}^2}{f\,(r_\text{o}^2+J^2)}(t-t_\text{o})\Delta-2EJ(t-t_\text{o})(\phi-\phi')$$

$$+\frac{E^2 r_{\rm o}^2}{f^2\left(r_{\rm o}^2+J^2\right)}\Delta^2+\left(r_{\rm o}^2+J^2\right)(\phi-\phi')^2+r_{\rm o}^2\left(\theta-\frac{\pi}{2}\right)^2 \tag{32}$$

with

$$\phi' \equiv \phi_{\rm o}-\frac{J\dot{r}\Delta}{f\left(r_{\rm o}^2+J^2\right)}, \tag{33}$$

where $\Delta \equiv r-r_{\rm o}$, and an identity $\dot{r}^2=E^2-f\left(1+J^2/r_{\rm o}^2\right)$ is used for simplifying the coefficient of Δ^2. Here, taking the coincidence limit $\Delta \to 0$, we have $\phi' \to \phi_{\rm o}$ (the same idea is found in Mino, Nakano, and Sasaki **(author?)** [11]). Also, in order to be multipole-decomposed globally, the quadratic part needs to be completely analytic and smooth over the entire two-sphere. For this purpose we rewrite it as

$$\begin{aligned}\mathcal{P}_{\rm II} &= (E^2-f)(t-t_{\rm o})^2-\frac{2E\dot{r}r_{\rm o}^2}{f\left(r_{\rm o}^2+J^2\right)}(t-t_{\rm o})\Delta-2EJ(t-t_{\rm o})\sin\theta\sin(\phi-\phi')\\ &\quad+\frac{E^2r_{\rm o}^2}{f^2\left(r_{\rm o}^2+J^2\right)}\Delta^2+(r_{\rm o}^2+J^2)\sin^2\theta\sin^2(\phi-\phi')+r_{\rm o}^2\cos^2\theta\\ &\quad+O[(x-x_{\rm o})^4],\end{aligned} \tag{34}$$

where one should notice that by replacing $\phi-\phi'=\sin(\phi-\phi')+O[(\phi-\phi')^3]$ and $1=\sin\theta+O[(\theta-\pi/2)^2]$ we create the $O[(x-x_{\rm o})^4]$ terms.

To aid in the multipole decomposition we rotate the usual Schwarzschild coordinates by following the approach of Barack and Ori **(author?)** [10] and Paper I **(author?)** [7] such that the coordinate location of the particle is moved from the equatorial plane $\theta=\frac{\pi}{2}$ to a location where $\sin\Theta=0$ (Θ: new polar angle). We define new angles Θ and Φ in terms of the usual Schwarzschild angles by

$$\begin{aligned}\sin\theta\cos(\phi-\phi') &= \cos\Theta\\ \sin\theta\sin(\phi-\phi') &= \sin\Theta\cos\Phi\\ \cos\theta &= \sin\Theta\sin\Phi.\end{aligned} \tag{35}$$

Also, under this coordinate rotation, a spherical harmonic $Y_{\ell m}(\theta,\phi)$ becomes

$$Y_{\ell m}(\theta,\phi)=\sum_{m'=-\ell}^{\ell}\alpha^{\ell}_{mm'}Y_{\ell m'}(\Theta,\Phi), \tag{36}$$

where the coefficients $\alpha^{\ell}_{mm'}$ depend on the rotation $(\theta,\phi)\to(\Theta,\Phi)$ as well as on ℓ, m, and m', and the index ℓ is preserved under the rotation **(author?)** [15]. As recognized already in Paper I **(author?)** [7], there is a great advantage of using the rotated angles (Θ,Φ): after expanding $\partial_a(q/\rho)$ into a sum of spherical harmonic components, we take the coincidence limit $\Delta\to 0$, $\Theta\to 0$. Then, finally only the $m=0$ components contribute to the self-force at $\Theta=0$ since $Y_{\ell m}(0,\Phi)=0$ for $m\neq 0$. Thus, the regularization parameters of Eq. (10) are just (ℓ, $m=0$) spherical harmonic components of $\partial_a(q/\rho)$ evaluated at $x^a_{\rm o}$.

Now, using these rotated angles, we may re-express Eq. (34) as

$$\begin{aligned}\mathcal{P}_{\rm II} &= (E^2-f)(t-t_{\rm o})^2-\frac{2E\dot{r}r_{\rm o}^2}{f\left(r_{\rm o}^2+J^2\right)}(t-t_{\rm o})\Delta-2EJ(t-t_{\rm o})\sin\Theta\cos\Phi\\ &\quad+2\left(r_{\rm o}^2+J^2\right)\left(1-\frac{J^2\sin^2\Phi}{r_{\rm o}^2+J^2}\right)\left[\frac{r_{\rm o}^2E^2\Delta^2}{2f^2\left(r_{\rm o}^2+J^2\right)^2\left(1-\frac{J^2\sin^2\Phi}{r_{\rm o}^2+J^2}\right)}+1-\cos\Theta\right]\\ &\quad+O[(x-x_{\rm o})^4],\end{aligned} \tag{37}$$

where an approximation $\sin^2\Theta=2(1-\cos\Theta)+O(\Theta^4)$ is used, the error from which is essentially $O(\Theta^4)=O[(x-x_{\rm o})^4]$ and can be absorbed into $\mathcal{P}_{\rm IV}$. Here one should note that through a series of modifications of the quadratic part of Eq. (29) we have created additional quartic order terms apart from the desired form. Then, we may remove these additional terms from the quadratic part and incorporate them into the quartic part $\mathcal{P}_{\rm IV}$ in Eq. (31). It is not necessary, however, to specify this quartic part: as already mentioned above, later in Subsection V C we will show that the quartic part $\mathcal{P}_{\rm IV}$ starts appearing from the ϵ^0-term and verify that the regularization parameters of the ϵ^0-term always vanish by analyzing the generic structure of the ϵ^0-term.

After removing $O[(x-x_{\rm o})^4]$ from Eq. (37) we may define

$$\begin{aligned}\tilde{\rho}^2 \;&\equiv\; (E^2-f)(t-t_{\rm o})^2 - \frac{2E\dot{r}r_{\rm o}^2}{f\left(r_{\rm o}^2+J^2\right)}(t-t_{\rm o})\Delta - 2EJ(t-t_{\rm o})\sin\Theta\cos\Phi \\ &+2\left(r_{\rm o}^2+J^2\right)\left(1-\frac{J^2\sin^2\Phi}{r_{\rm o}^2+J^2}\right)\left[\frac{r_{\rm o}^2E^2\Delta^2}{2f^2\left(r_{\rm o}^2+J^2\right)^2\left(1-\frac{J^2\sin^2\Phi}{r_{\rm o}^2+J^2}\right)}+1-\cos\Theta\right]. \end{aligned} \tag{38}$$

In particular, when fixing $t=t_{\rm o}$, we define

$$\tilde{\rho}_{\rm o}^2 \equiv \tilde{\rho}^2\big|_{t=t_{\rm o}} = 2\left(r_{\rm o}^2+J^2\right)\chi\left(\delta^2+1-\cos\Theta\right) \tag{39}$$

with

$$\chi \equiv 1-\frac{J^2\sin^2\Phi}{r_{\rm o}^2+J^2} \tag{40}$$

and

$$\delta^2 \equiv \frac{r_{\rm o}^2E^2\Delta^2}{2f^2\left(r_{\rm o}^2+J^2\right)^2\chi}. \tag{41}$$

Now we rewrite Eq. (31) by replacing the original quadratic part $\mathcal{P}_{\rm II}$ with the modified form $\tilde{\rho}^2$ above,

$$\rho^2 = \epsilon^2\tilde{\rho}^2 + \epsilon^3\mathcal{P}_{\rm III} + \epsilon^4\mathcal{P}_{\rm IV} + O(\epsilon^5), \tag{42}$$

where the new quartic part $\mathcal{P}_{\rm IV}$ includes the additional quartic order terms that result from modification of the quadratic part $\mathcal{P}_{\rm II}$. Then, based on this redefined ρ^2, we have the following expression of $\partial_a(1/\rho)|_{t=t_{\rm o}}$ in a Laurent series expansion

$$\partial_a\left(\frac{1}{\rho}\right)\bigg|_{t=t_{\rm o}} = -\frac{1}{2}\frac{\partial_a\left(\tilde{\rho}^2\right)\big|_{t=t_{\rm o}}}{\tilde{\rho}_{\rm o}^3}\epsilon^{-2} + \left\{-\frac{1}{2}\frac{\partial_a\mathcal{P}_{\rm III}\big|_{t=t_{\rm o}}}{\tilde{\rho}_{\rm o}^3} + \frac{3}{4}\frac{\left[\partial_a\left(\tilde{\rho}^2\right)\right]\mathcal{P}_{\rm III}\big|_{t=t_{\rm o}}}{\tilde{\rho}_{\rm o}^5}\right\}\epsilon^{-1} + O(\epsilon^0). \tag{43}$$

Eq. (39), when inserted into Eq. (43), plays a very important role in calculating the regularization parameters for the rest of the section: out of Eq. (39), we induce Legendre polynomial expansions in terms of $\cos\Theta$. Sometimes we may have the dependence on $\tilde{\rho}_{\rm o}^2$ not only in the denominators but also in the numerators on the right hand side of Eq. (43). In the numerators the dependence can be found from the terms containing $\sin^n\Theta$ or $\cos^n\Theta$ since Eq. (39) can be solved for $\cos\Theta$. After finding all of the $\tilde{\rho}_{\rm o}^2$ dependence, the rest of the calculations involve integrating over the angle Φ . The techniques involved in Legendre polynomial expansions and integration over Φ are described in detail in Appendices C and D of Paper I **(author?)** [7].

Below in Subsections V A and V B, we present the key steps of calculating the regularization parameters.

A. A_a-terms

We take the ϵ^{-2} term from Eq. (43) and define

$$Q_a[\epsilon^{-2}] \equiv -\frac{q^2}{2}\frac{\partial_a\left(\tilde{\rho}^2\right)\big|_{t=t_{\rm o}}}{\tilde{\rho}_{\rm o}^3} \tag{44}$$

Then, we proceed with our calculations of the regularization parameters one component at a time.

1. *A_t-term:*

First we complete the expression for $Q_t[\epsilon^{-2}]$ by recalling Eqs. (38) and (39)

$$Q_t[\epsilon^{-2}] \;=\; -\frac{q^2}{2}\tilde{\rho}_{\rm o}^{-3}\,\partial_t\left(\tilde{\rho}^2\right)\big|_{t=t_{\rm o}}$$

$$
\begin{aligned}
&= \frac{q^2}{2}\left[2\left(r_{\mathrm{o}}^2+J^2\right)\chi\left(\delta^2+1-\cos\Theta\right)\right]^{-3/2}\left(\frac{2E\dot{r}r_{\mathrm{o}}^2\Delta}{f\left(r_{\mathrm{o}}^2+J^2\right)}+2EJ\sin\Theta\cos\Phi\right)\\
&= \frac{q^2E\dot{r}r_{\mathrm{o}}^2\Delta\chi^{-3/2}}{2\sqrt{2}f\left(r_{\mathrm{o}}^2+J^2\right)^{5/2}}\left(\delta^2+1-\cos\Theta\right)^{-3/2}\\
&\quad -\frac{q^2EJ\chi^{-3/2}\cos\Phi}{\sqrt{2}\left(r_{\mathrm{o}}^2+J^2\right)^{3/2}}\left.\frac{\partial}{\partial\Theta}\right|_{\Delta}\left(\delta^2+1-\cos\Theta\right)^{-1/2}, \qquad (45)
\end{aligned}
$$

where $\left.\frac{\partial}{\partial\Theta}\right|_{\Delta}$ means that Δ is held constant while the differntiation is performed with respect to Θ.

According to Appendix D of Paper I **(author?)** [7], for $p \geq 1$

$$
\left(\delta^2+1-\cos\Theta\right)^{-p-1/2}=\sum_{\ell=0}^{\infty}\frac{2\ell+1}{\delta^{2p-1}(2p-1)}\left[1+O(\ell\delta)\right]P_\ell(\cos\Theta),\ \delta\to 0, \qquad (46)
$$

and for $p=0$

$$
\left(\delta^2+1-\cos\Theta\right)^{-1/2}=\sum_{\ell=0}^{\infty}\left[\sqrt{2}+O(\ell\delta)\right]P_\ell(\cos\Theta),\ \delta\to 0. \qquad (47)
$$

Then, by Eqs. (46) for $p=1$, (47), and (41), in the limit $\delta\to 0$ (equivalently $\Delta\to 0$) Eq. (45) becomes

$$
\begin{aligned}
\lim_{\Delta\to 0}Q_t[\epsilon^{-2}] &= \mathrm{sgn}(\Delta)\frac{q^2\dot{r}r_{\mathrm{o}}\chi^{-1}}{\left(r_{\mathrm{o}}^2+J^2\right)^{3/2}}\sum_{\ell=0}^{\infty}\left(\ell+\frac{1}{2}\right)P_\ell(\cos\Theta)\\
&\quad -\frac{q^2EJ\chi^{-3/2}\cos\Phi}{\left(r_{\mathrm{o}}^2+J^2\right)^{3/2}}\sum_{\ell=0}^{\infty}\left.\frac{\partial}{\partial\Theta}\right|_{\Delta}P_\ell(\cos\Theta). \qquad (48)
\end{aligned}
$$

Then, we integrate $\lim_{\Delta\to 0}Q_t[\epsilon^{-2}]$ over Φ and divide it by 2π (we denote this process by the angle brackets "$\langle\rangle$")

$$
\left\langle\lim_{\Delta\to 0}Q_t[\epsilon^{-2}]\right\rangle=\mathrm{sgn}(\Delta)\frac{q^2\dot{r}r_{\mathrm{o}}\left\langle\chi^{-1}\right\rangle}{\left(r_{\mathrm{o}}^2+J^2\right)^{3/2}}\sum_{\ell=0}^{\infty}\left(\ell+\frac{1}{2}\right)P_\ell(\cos\Theta), \qquad (49)
$$

where we exploit the fact that $\left\langle\chi^{-3/2}\cos\Phi\right\rangle=0$ to get rid of the second part in Eq. (48) [18]. Appendix C of Paper I **(author?)** [7] provides $\left\langle\chi^{-1}\right\rangle={}_2F_1\left(1,\frac{1}{2};1;\alpha\right)\equiv F_1=(1-\alpha)^{-1/2}$, where $\alpha\equiv J^2/\left(r_{\mathrm{o}}^2+J^2\right)$. Substituting this into Eq. (49), the regularization parameter A_t can be finally determined when we take the coefficient of the sum on the right hand side in the coincidence limit $\Theta\to 0$

$$
A_t=\mathrm{sgn}(\Delta)\frac{q^2}{r_{\mathrm{o}}^2}\frac{\dot{r}}{1+J^2/r_{\mathrm{o}}^2}. \qquad (50)
$$

2. A_r-term:

Similarly, we have

$$
Q_r[\epsilon^{-2}]=-\frac{q^2}{2}\tilde{\rho}_{\mathrm{o}}^{-3}\,\partial_r\left(\tilde{\rho}^2\right)\Big|_{t=t_{\mathrm{o}}}. \qquad (51)
$$

Here, before computing $\partial_r\left(\tilde{\rho}^2\right)\big|_{t=t_{\mathrm{o}}}$ we reverse the process via Eqs. (32), (34), (37), and (38) to obtain the relation

$$
\begin{aligned}
\tilde{\rho}^2 &= \mathcal{P}_{\mathrm{II}}+O[(x-x_{\mathrm{o}})^4]\\
&= (E^2-f)(t-t_{\mathrm{o}})^2-\frac{2E\dot{r}r_{\mathrm{o}}^2}{f\left(r_{\mathrm{o}}^2+J^2\right)}(t-t_{\mathrm{o}})\Delta-2EJ(t-t_{\mathrm{o}})(\phi-\phi') \qquad (52)\\
&\quad +\frac{E^2r_{\mathrm{o}}^2}{f^2\left(r_{\mathrm{o}}^2+J^2\right)}\Delta^2+\left(r_{\mathrm{o}}^2+J^2\right)(\phi-\phi')^2+r_{\mathrm{o}}^2\left(\theta-\frac{\pi}{2}\right)^2\\
&\quad +O[(x-x_{\mathrm{o}})^4]. \qquad (53)
\end{aligned}
$$

Differentiating this with respect to r and going through the process via Eqs. (34) and (35), Eq. (51) can be expressed with the help of Eq. (39) as

$$Q_r[\epsilon^{-2}] = -\frac{q^2}{f^2}\left[2\left(r_\text{o}^2+J^2\right)\chi\left(\delta^2+1-\cos\Theta\right)\right]^{-3/2}\left[\frac{r_\text{o}^2E^2\Delta}{r_\text{o}^2+J^2}+fJ\dot{r}\sin\Theta\cos\Phi\right] \tag{54}$$

[19]. Then, the rest of the calculation is carried out in the same fashion as for the case of A_t-term above. We obtain

$$A_r = -\text{sgn}(\Delta)\frac{q^2}{r_\text{o}^2}\frac{E}{f\left(1+J^2/r_\text{o}^2\right)}. \tag{55}$$

3. A_ϕ-term:

First we have

$$Q_\phi[\epsilon^{-2}] \;=\; -\frac{q^2}{2}\tilde{\rho}_\text{o}^{-3}\,\partial_\phi\left(\tilde{\rho}^2\right)\big|_{t=t_\text{o}}. \tag{56}$$

Taking the same steps as used for A_r-term above via Eqs. (53), (34), and (35) in order, we obtain

$$\partial_\phi\left(\tilde{\rho}^2\right)\big|_{t=t_\text{o}} = 2\left(r_\text{o}^2+J^2\right)\sin\Theta\cos\Phi + O[(x-x_\text{o})^3]. \tag{57}$$

Then, in a similar manner to that employed in the previous cases, in the limit $\Delta\to 0$ Eq. (56) becomes

$$\lim_{\Delta\to 0}Q_\phi[\epsilon^{-2}] = -\frac{q^2\chi^{-3/2}\cos\Phi}{\left(r_\text{o}^2+J^2\right)^{1/2}}\sum_{\ell=0}^{\infty}\frac{\partial}{\partial\Theta}\bigg|_\Delta P_\ell(\cos\Theta) \tag{58}$$

[20]. The right hand side vanishes through "$\langle\,\rangle$" process because $\left\langle\chi^{-3/2}\cos\Phi\right\rangle = 0$. Hence,

$$A_\phi = 0. \tag{59}$$

4. A_θ-term:

It is evident from the particle's motion, which is confined to the equatorial plane $\theta_\text{o} = \frac{\pi}{2}$, that no self force is acting on the particle in the direction perpendicular to this plane. This is due to the fact that both the derivatives of retarded field and the singular source field with respect to θ tend to zero in the coincidence limit. Our calculation of A_θ should support this. Through the same process as employed before, we have

$$Q_\theta[\epsilon^{-2}] = -\frac{q^2}{2}\tilde{\rho}_\text{o}^{-3}\,\partial_\theta\left(\tilde{\rho}^2\right)\big|_{t=t_\text{o}} \tag{60}$$

with

$$\partial_\theta\left(\tilde{\rho}^2\right)\big|_{t=t_\text{o}} = 2r_\text{o}^2\sin\Theta\sin\Phi + O[(x-x_\text{o})^3]. \tag{61}$$

Then, similarly as in the case of A_ϕ-term above

$$\lim_{\Delta\to 0}Q_\theta[\epsilon^{-2}] = -\frac{q^2r_\text{o}^2\chi^{-3/2}\sin\Phi}{\left(r_\text{o}^2+J^2\right)^{3/2}}\sum_{\ell=0}^{\infty}\frac{\partial}{\partial\Theta}\bigg|_\Delta P_\ell(\cos\Theta). \tag{62}$$

Again, via "$\langle\,\rangle$" process, the right hand side vanishes because $\left\langle\chi^{-3/2}\sin\Phi\right\rangle = 0$. Thus,

$$A_\theta = 0. \tag{63}$$

B. B_a-terms

We take the ϵ^{-1} term from Eq. (43) and define

$$Q_a[\epsilon^{-1}] \equiv q^2 \left\{ -\frac{1}{2} \frac{\partial_a \mathcal{P}_{\rm III}\big|_{t=t_{\rm o}}}{\tilde{\rho}_{\rm o}^3} + \frac{3}{4} \frac{\left[\partial_a \left(\tilde{\rho}^2\right)\right] \mathcal{P}_{\rm III}\big|_{t=t_{\rm o}}}{\tilde{\rho}_{\rm o}^5} \right\}, \tag{64}$$

where for computing $\partial_a \left(\tilde{\rho}^2\right)$, Eq. (53) should be referred to, and $\mathcal{P}_{\rm III}$ is the cubic part taken directly from Eq. (29).

We may express this in a generic form

$$Q_a[\epsilon^{-1}] = \sum_{n=1}^{2} \sum_{k=0}^{2n} \sum_{p=0}^{[k/2]} \frac{b_{nkp(a)} \Delta^{2n-k} \left(\phi - \phi_{\rm o}\right)^{k-2p} \left(\theta - \frac{\pi}{2}\right)^{2p}}{\tilde{\rho}_{\rm o}^{2n+1}}, \tag{65}$$

where $\Delta \equiv r - r_{\rm o}$, and $b_{nkp(a)}$ is the coefficient of each individual term that depends on n, k and p as well as a, with a dimension $\mathcal{R}^{k-1}$ for $a = t$, r and $\mathcal{R}^k$ for $a = \theta$, ϕ. We recall from Eqs. (32) and (33) that the first of the steps to lead to $\tilde{\rho}_{\rm o}^2$ in the denominator is replacing $\phi - \phi_{\rm o}$ by $(\phi - \phi') - \frac{J\dot{r}}{f(r_{\rm o}^2+J^2)}\Delta$ to eliminate the coupling term $\Delta\left(\phi - \phi_{\rm o}\right)$. This makes a sum of independent square forms of each of Δ and $\phi - \phi'$, which is a necessary step to induce the Legendre polynomial expansions later. Thus, to be consistent with this modification in the denominator, $\phi - \phi_{\rm o}$ in the numerator on the right hand side of Eq. (65) should be also replaced by $(\phi - \phi') - \frac{J\dot{r}}{f(r_{\rm o}^2+J^2)}\Delta$. Then, this will create a number of additional terms apart from $(\phi - \phi')^m$ when we expand the quantity $\left[(\phi - \phi') - \frac{J\dot{r}}{f(r_{\rm o}^2+J^2)}\Delta\right]$ raised, say, to the m-th power, and the computation will be very complicated.

By analyzing the structure of the quantity on the right hand side of Eq. (65) one can prove that $\phi - \phi_{\rm o}$ may be replaced just by $\phi - \phi'$ in the numerator without the term $-\frac{J\dot{r}}{f(r_{\rm o}^2+J^2)}\Delta$ (the same idea is found in Mino, Nakano, and Sasaki **(author?)** [11]). The verification follows. The behavior of the quantity on the right hand side of Eq. (65), according to the powers of each factor, is

$$Q_a[\epsilon^{-1}] \sim \tilde{\rho}_{\rm o}^{-(2n+1)} \Delta^{2n-k} \left(\phi - \phi_{\rm o}\right)^{k-2p} \left(\theta - \frac{\pi}{2}\right)^{2p} \mathcal{R}^s, \tag{66}$$

where $s = k - 1$ for $a = t$, r and $s = k$ for $a = \theta$, ϕ. Further,

$$\begin{aligned}
\left(\phi - \phi_{\rm o}\right)^{k-2p} &= \left[(\phi - \phi') - \frac{J\dot{r}\Delta}{f(r_{\rm o}^2 + J^2)}\right]^{k-2p} \\
&= \sum_{i=0}^{k-2p} c_{kpi} \left(\phi - \phi'\right)^i \Delta^{k-2p-i} \sim \left(\phi - \phi'\right)^i \Delta^{k-2p-i} / \mathcal{R}^{k-2p-i} \qquad (67) \\
&\sim \left(\sin\Theta\right)^i \left(\cos\Phi\right)^i \Delta^{k-2p-i} / \mathcal{R}^{k-2p-i} + O[(x - x_{\rm o})^{k-2p+2}], \qquad (68)
\end{aligned}$$

where a binomial expansion over the index $i = 0, 1, \cdots, k - 2p$ is assumed with $c_{kpi} \sim 1/\mathcal{R}^{k-2p-i}$ in Eq. (67), and in Eq. (68) $\left(\phi - \phi'\right)^i$ is replaced by $[\sin(\phi - \phi')]^i + O[\left(\phi - \phi'\right)^{i+2}]$ — the term $O[(x - x_{\rm o})^{k-2p+2}]$ at the end results from this $O[\left(\phi - \phi'\right)^{i+2}]$, then the coordinates are rotated using the definition of new angles by Eq. (35). Also, by Eq. (35) again

$$\left(\theta - \frac{\pi}{2}\right)^{2p} = \left(\sin\Theta\right)^{2p} \left(\sin\Phi\right)^{2p} + O[(x - x_{\rm o})^{2p+2}]. \tag{69}$$

Using Eqs. (68) and (69), the behavior of $Q[\epsilon^{-1}]$ in Eq. (66) looks like

$$Q_a[\epsilon^{-1}] \sim \tilde{\rho}_{\rm o}^{-(2n+1)} \Delta^{2n-2p-i} \left(\sin\Theta\right)^{2p+i} \left(\cos\Phi\right)^i \left(\sin\Phi\right)^{2p} \mathcal{R}^s, \tag{70}$$

where $s = 2p + i - 1$ for $a = t$, r and $s = 2p + i$ for $a = \theta$, ϕ, and any contributions from $O[(x - x_{\rm o})^{k-2p+2}]$ in Eq. (68) and from $O[(x - x_{\rm o})^{2p+2}]$ in Eq. (69) have been disregarded: by putting these pieces into Eq. (66) we simply obtain ϵ^1-terms, which would correspond to $O(\ell^{-2})$ in Eq. (10) and should vanish when summed over ℓ in our final self-force calculation by Eq. (9) [21]. $Q_a[\epsilon^{-1}]$ then can be categorized into the following cases:

(i) $i = 2j+1$ ($j = 0,\, 1,\, 2,\, \cdots$)
The integrand for "$\langle\,\rangle$" process, $F(\Phi) \equiv (\cos\Phi)^{2j+1}\,(\sin\Phi)^{2p}$ has the property $F(\Phi+\pi) = -F(\Phi)$. Thus

$$\left\langle Q_a[\epsilon^{-1}]\right\rangle = 0, \tag{71}$$

(ii) $i = 2j$ ($j = 0,\, 1,\, 2,\, \cdots$)
Using Eqs. (39) and (41), we can express $(\sin\Theta)^{2p+i}$ in Eq. (70) above in terms of $\tilde{\rho}_{\rm o}$ and Δ via a binomial expansion

$$\begin{aligned}(\sin\Theta)^{2p+2j} &= [2\,(1-\cos\Theta)]^{p+j} + O[(x-x_{\rm o})^{2(p+j)+2}] \\ &= \sum_{q=0}^{p+j} d_{pjq}\tilde{\rho}_{\rm o}^{2q}\Delta^{2(p+j-q)} + O[(x-x_{\rm o})^{2(p+j)+2}] \qquad (72)\\ &\sim \tilde{\rho}_{\rm o}^{2q}\Delta^{2(p+j-q)}/\mathcal{R}^{2(p+j)} + O[(x-x_{\rm o})^{2(p+j)+2}], \qquad (73)\end{aligned}$$

where $q = 0,\, 1,\, \cdots,\, p+j$ is the index for a binomial expansion and $d_{pjq} \sim 1/\mathcal{R}^{2(p+j)}$. When Eq. (73) is substituted into Eq. (70), the contribution from $O[(x-x_{\rm o})^{2(p+j)+2}]$ can be disregarded since it would correspond to $O(\epsilon^1)$ again. Then, we have

$$Q_a[\epsilon^{-1}] \sim (\sin\Phi)^{2p}\,(\cos\Phi)^{2j}\,\tilde{\rho}_{\rm o}^{-2(n-q)-1}\Delta^{2(n-q)}\mathcal{R}^s, \tag{74}$$

where $s = -1$ for $a = t,\, r$ and $s = 0$ for $a = \theta,\, \phi$, and we can guarantee that $n - q \geq 0$ always since $0 \leq q \leq p+j = p+\frac{1}{2}i$, $0 \leq i \leq k-2p$, and $p \leq k \leq 2n$. Then, Eq. (74) can be subcategorized into the following two cases;

(ii-1) $n - q \geq 1$
By Eqs. (39), (41), and (46)

$$Q_a[\epsilon^{-1}] \underset{\Delta\to 0}{\sim} (\sin\Phi)^{2p}\,(\cos\Phi)^{2j}\,\Delta P_\ell(\cos\Theta)\mathcal{R}^s \longrightarrow 0, \tag{75}$$

(ii-2) $n - q = 0$
By Eqs. (39), (41), and (47)

$$Q_a[\epsilon^{-1}] \underset{\Delta\to 0}{\sim} (\sin\Phi)^{2p}\,(\cos\Phi)^{2j}\,P_\ell(\cos\Theta)\mathcal{R}^s, \tag{76}$$

where $s = -1$ for $a = t,\, r$ and $s = 0$ for $a = \theta,\, \phi$.

Therefore, by analyzing the structure of $Q_a[\epsilon^{-1}]$ we find that the ϵ^{-1}-terms vanish in all the cases except when $n - q = 0$. The non-vanishing B_a-terms are derived only from this case. Then, by $0 \leq q \leq p+j = p+\frac{1}{2}i$, $0 \leq i \leq k-2p$, and $p \leq k \leq 2n$ together with $n = q$ one can show that

$$0 \;\leq\; k-2p-i \text{ and } k-2p-i \leq 0, \text{ i.e. } k-2p-i = 0. \tag{77}$$

Substituting this result into Eq. (67), then into Eq. (65) we may conclude that in the numerator of $Q[\epsilon^{-1}]$ in Eq. (65) one can simply substitute

$$(\phi-\phi_{\rm o})^{k-2p} \to (\phi-\phi')^{k-2p}\,.\ \text{Q. E. D.} \tag{78}$$

The significance of this proof does not lie in the result given by Eq. (78) only, but also in the fact that the non-vanishing contribution comes only from the case $n = q$ for Eq. (74), i.e.

$$Q_a[\epsilon^{-1}] \sim (\sin\Phi)^{2p}\,(\cos\Phi)^{2(n-p)}\,\tilde{\rho}_{\rm o}^{-1}\mathcal{R}^s, \tag{79}$$

where $n = 1,\, 2$ and $0 \leq p \leq n$, and $s = -1$ for $a = t,\, r$ and $s = 0$ for $a = \theta,\, \phi$.

Below are presented the calculations of B_a-terms of the regularization parameters by component, in a similar manner to those for A_a-terms .

1. B_t-term:

We begin with

$$Q_t[\epsilon^{-1}] = q^2 \left\{ -\frac{1}{2} \frac{\partial_t \mathcal{P}_{\rm III}|_{t=t_o}}{\tilde{\rho}_o^3} + \frac{3}{4} \frac{\left[\partial_t \left(\tilde{\rho}^2\right)\right] \mathcal{P}_{\rm III}|_{t=t_o}}{\tilde{\rho}_o^5} \right\}. \tag{80}$$

The subsequent computation will be very lengthy and it will be reasonable to split $Q_t[\epsilon^{-1}]$ into two parts. First, let

$$Q_{t(1)}[\epsilon^{-1}] \equiv -\frac{q^2}{2} \tilde{\rho}_o^{-3}\, \partial_t \mathcal{P}_{\rm III}|_{t=t_o}\,, \tag{81}$$

where

$$\partial_t \mathcal{P}_{\rm III}|_{t=t_o} = -\frac{ME\dot{r}\Delta^2}{f^2 r_o^2} - 2\left(1 - \frac{M}{r_o}\right) \frac{EJ\Delta}{f r_o} (\phi - \phi_o) + r_o E\dot{r} \left[(\phi - \phi_o)^2 + \left(\theta - \frac{\pi}{2}\right)^2 \right]. \tag{82}$$

As proved at the beginning of this Subsection, every $(\phi - \phi_o)^m$ in the numerators of the ϵ^{-1}-term can be replaced by $(\phi - \phi')^m$ without affecting the rest of calculation. Then, followed by the rotation of the coordinates via Eq. (35)

$$\begin{aligned} Q_{t(1)}[\epsilon^{-1}] \;=\; & -\frac{q^2}{2} \tilde{\rho}_o^{-3} \left[-\frac{ME\dot{r}\Delta^2}{f^2 r_o^2} - 2\left(1 - \frac{M}{r_o}\right) \frac{EJ\Delta}{f r_o} \sin\Theta \cos\Phi + 2 r_o E\dot{r} \left(1 - \cos\Theta\right) \right] \\ & + O\left[\frac{(x - x_o)^4}{\tilde{\rho}_o^3}\right], \end{aligned} \tag{83}$$

where an approximation $\sin^2\Theta = 2(1 - \cos\Theta) + O[(x - x_o)^4]$ is used to obtain the last term inside the first bracket. Here we may drop off the term $O\left[(x - x_o)^4/\tilde{\rho}_o^3\right]$, which is essentially $O(\epsilon^1)$, for the same reason as explained at the beginning of this subsection. Then, using the same techniques as used to find A_a-terms, we can reduce Eq. (83) to

$$\begin{aligned} Q_{t(1)}[\epsilon^{-1}] \;=\; & \left[\frac{q^2 ME\dot{r}}{2 f^2 r_o^2} + \frac{q^2 r_o^3 E^3 \dot{r} \chi^{-1}}{2 f^2 \left(r_o^2 + J^2\right)^2} \right] \Delta^2 \left[2\left(r_o^2 + J^2\right) \chi \left(\delta^2 + 1 - \cos\Theta\right)\right]^{-3/2} \\ & - \frac{q^2 \left(1 - \frac{M}{r_o}\right) EJ\Delta \chi^{-3/2} \cos\Phi}{\sqrt{2} f r_o \left(r_o^2 + J^2\right)^{3/2}} \left.\frac{\partial}{\partial\Theta}\right|_{\Delta} \left(\delta^2 + 1 - \cos\Theta\right)^{-1/2} - \frac{q^2 E\dot{r} r_o \chi^{-1}}{2\left(r_o^2 + J^2\right)} \tilde{\rho}_o^{-1}. \end{aligned} \tag{84}$$

As we have seen before, by Eq. (46) $\left(\delta^2 + 1 - \cos\Theta\right)^{-3/2} \sim \Delta^{-1}$ in the limit $\Delta \to 0$ and the first term on the right hand side will vanish. The second term will also give no contribution to the regularization parameters because $\left\langle \chi^{-3/2} \cos\Phi \right\rangle = 0$. Only the last term, which is $\sim \tilde{\rho}_o^{-1}$, will give non-zero contribution according to the argument in the analysis presented above (see Eq. (79)). Using Eq. (47) in the limit $\Delta \to 0$ and taking "$\langle\rangle$" process, Eq. (84) becomes

$$\left\langle \lim_{\Delta \to 0} Q_{t(1)}[\epsilon^{-1}] \right\rangle = -\frac{1}{2} \frac{q^2}{r_o^2} \frac{E\dot{r} \left\langle \chi^{-3/2} \right\rangle}{\left(1 + J^2/r_o^2\right)^{3/2}} \sum_{\ell=0}^{\infty} P_\ell\left(\cos\Theta\right). \tag{85}$$

The identity $\langle \chi^{-p} \rangle \equiv \left\langle \left(1 - \alpha \sin^2\Phi\right)^{-p} \right\rangle = {}_2F_1\left(p,\, \frac{1}{2};\, 1,\, \alpha\right) \equiv F_p$, with $\alpha \equiv J^2/\left(r_o^2 + J^2\right)$ is taken from Appendix C of Paper I **(author?)** [7], and we take the limit $\Theta \to 0$

$$\left.\left\langle \lim_{\Delta \to 0} Q_{t(1)}[\epsilon^{-1}] \right\rangle\right|_{\Theta \to 0} = -\frac{1}{2} \frac{q^2}{r_o^2} \frac{E\dot{r} F_{3/2}}{\left(1 + J^2/r_o^2\right)^{3/2}}. \tag{86}$$

Now the remaining part is

$$Q_{t(2)}[\epsilon^{-1}] \equiv \frac{3q^2}{4} \tilde{\rho}_o^{-5} \left[\partial_t \left(\tilde{\rho}^2\right)\right] \mathcal{P}_{\rm III}|_{t=t_o}\,, \tag{87}$$

where

$$
\begin{aligned}
\left[\partial_t\left(\tilde{\rho}^2\right)\right]\mathcal{P}_{\rm III}\big|_{t=t_{\rm o}} \;=\; & \left[-\frac{2E\dot{r}\Delta}{f}-2EJ\left(\phi-\phi_{\rm o}\right)\right] \\
& \times\left[-\left(1+\frac{\dot{r}^2}{f}\right)\frac{M\Delta^3}{f^2r_{\rm o}^2}+\left(2-\frac{5M}{r_{\rm o}}\right)\frac{J\dot{r}\Delta^2}{f^2r_{\rm o}}\left(\phi-\phi_{\rm o}\right)\right. \\
& +\left(1-\frac{\dot{r}}{f}+\frac{2J^2}{r_{\rm o}^2}\right)r_{\rm o}\Delta\left(\phi-\phi_{\rm o}\right)^2+\left(1-\frac{\dot{r}^2}{f}\right)r_{\rm o}\Delta\left(\theta-\frac{\pi}{2}\right)^2 \\
& \left.-r_{\rm o}J\dot{r}\left(\phi-\phi_{\rm o}\right)^3-r_{\rm o}J\dot{r}\left(\phi-\phi_{\rm o}\right)\left(\theta-\frac{\pi}{2}\right)^2\right]+O[(x-x_{\rm o})^6]. \qquad (88)
\end{aligned}
$$

Taking similar procedures as above, the non-vanishing contributions turn out to be

$$
\begin{aligned}
\left\langle\lim_{\Delta\to0}Q_{t(2)}[\epsilon^{-1}]\right\rangle \;=\; & \left\langle\lim_{\Delta\to0}\frac{3}{2}q^2EJ^2\dot{r}r_{\rm o}\tilde{\rho}_{\rm o}^{-5}\cos^2\Phi\sin^4\Theta\right\rangle \\
=\; & \left\langle\lim_{\Delta\to0}\frac{3}{2}\frac{q^2}{r_{\rm o}}\frac{E\dot{r}\tilde{\rho}_{\rm o}^{-1}}{1+J^2/r_{\rm o}^2}\left(\chi^{-1}-\frac{\chi^{-2}}{1+J^2/r_{\rm o}^2}\right)\right\rangle \\
=\; & \frac{3}{2}\frac{q^2}{r_{\rm o}^2}\frac{E\dot{r}}{\left(1+J^2/r_{\rm o}^2\right)^{3/2}}\left(\left\langle\chi^{-3/2}\right\rangle-\frac{\left\langle\chi^{-5/2}\right\rangle}{1+J^2/r_{\rm o}^2}\right)\sum_{\ell=0}^{\infty}P_\ell\left(\cos\Theta\right), \qquad (89)
\end{aligned}
$$

where all other terms than $\sim\tilde{\rho}_{\rm o}^{-1}$ again have been dropped off during the procedure since they vanish either in the limit $\Delta\to0$ or through the "$\langle\rangle$" process. Then, using the identity $\langle\chi^{-p}\rangle\equiv{}_2F_1\left(p,\,\frac{1}{2};\,1,\,\alpha\right)\equiv F_p$, we have

$$
\left\langle\lim_{\Delta\to0}Q_{t(2)}[\epsilon^{-1}]\right\rangle\Big|_{\Theta\to0}=\frac{3}{2}\frac{q^2}{r_{\rm o}^2}\frac{E\dot{r}}{\left(1+J^2/r_{\rm o}^2\right)^{3/2}}\left(F_{3/2}-\frac{F_{5/2}}{1+J^2/r_{\rm o}^2}\right). \qquad (90)
$$

By combining Eqs. (86) and (90), we finally obtain

$$
B_t=\frac{q^2}{r_{\rm o}^2}E\dot{r}\left[\frac{F_{3/2}}{\left(1+J^2/r_{\rm o}^2\right)^{3/2}}-\frac{3F_{5/2}}{2\left(1+J^2/r_{\rm o}^2\right)^{5/2}}\right]. \qquad (91)
$$

2. B_r-term:

From Eq. (64) we start with

$$
Q_r[\epsilon^{-1}]=q^2\left\{-\frac{1}{2}\frac{\partial_r\mathcal{P}_{\rm III}|_{t=t_{\rm o}}}{\tilde{\rho}_{\rm o}^3}+\frac{3}{4}\frac{\left[\partial_r\left(\tilde{\rho}^2\right)\right]\mathcal{P}_{\rm III}\big|_{t=t_{\rm o}}}{\tilde{\rho}_{\rm o}^5}\right\}. \qquad (92)
$$

Then, following the same steps as taken for the case of B_t-term above, we obtain

$$
B_r=\frac{q^2}{r_{\rm o}^2}\left[-\frac{F_{1/2}}{\left(1+J^2/r_{\rm o}^2\right)^{1/2}}+\frac{(1-2f^{-1}\dot{r}^2)F_{3/2}}{2\left(1+J^2/r_{\rm o}^2\right)^{3/2}}+\frac{3f^{-1}\dot{r}^2F_{5/2}}{2\left(1+J^2/r_{\rm o}^2\right)^{5/2}}\right]. \qquad (93)
$$

3. B_ϕ-term:

Again, from Eq. (64)

$$
Q_\phi[\epsilon^{-1}]=q^2\left\{-\frac{1}{2}\frac{\partial_\phi\mathcal{P}_{\rm III}|_{t=t_{\rm o}}}{\tilde{\rho}_{\rm o}^3}+\frac{3}{4}\frac{\left[\partial_\phi\left(\tilde{\rho}^2\right)\right]\mathcal{P}_{\rm III}\big|_{t=t_{\rm o}}}{\tilde{\rho}_{\rm o}^5}\right\}. \qquad (94)
$$

Then, similarly we can derive

$$
B_\phi=\frac{q^2}{J}\dot{r}\left[\frac{F_{1/2}-F_{3/2}}{\left(1+J^2/r_{\rm o}^2\right)^{1/2}}+\frac{3(F_{5/2}-F_{3/2})}{2\left(1+J^2/r_{\rm o}^2\right)^{3/2}}\right]. \qquad (95)
$$

4. B_θ-term:

As A_θ vanishes, so should B_θ. From

$$Q_\theta[\epsilon^{-1}] = q^2 \left\{ -\frac{1}{2}\frac{\partial_\theta \mathcal{P}_{\rm III}|_{t=t_o}}{\tilde{\rho}_o^3} + \frac{3}{4}\frac{\left[\partial_\theta\left(\tilde{\rho}^2\right)\right]\mathcal{P}_{\rm III}|_{t=t_o}}{\tilde{\rho}_o^5} \right\}, \tag{96}$$

one finds that there is no term like $\sim \tilde{\rho}_o^{-1}$: all terms are either like $\sim \Delta^{2n}/\tilde{\rho}_o^{2n+1}$ or like $\sim \Delta^{2n-1}\sin\Theta\cos\Phi/\tilde{\rho}_o^{2n+1}$ ($n = 1,\, 2$), which vanish in the limit $\Delta \to 0$ or through the "$\langle\,\rangle$" process. Thus

$$B_\theta = 0. \tag{97}$$

C. C_a-terms

We have mentioned before that C_a-terms, which originate from ϵ^0-term in Eq. (43), always vanish. This can be proved by analyzing the structure of ϵ^0-term. First we specify the ϵ^0-order term for $\partial_a(1/\rho)|_{t=t_o}$ in a Laurent series expansion and define

$$Q_a[\epsilon^0] \equiv q^2 \left\{ -\frac{1}{2}\frac{\partial_a \mathcal{P}_{\rm IV}|_{t=t_o}}{\tilde{\rho}_o^3} + \frac{3}{4}\frac{\left(\partial_a\mathcal{P}_{\rm III}\right)\mathcal{P}_{\rm III}|_{t=t_o} + \left[\partial_a\left(\tilde{\rho}^2\right)\right]\mathcal{P}_{\rm IV}|_{t=t_o}}{\tilde{\rho}_o^5} - \frac{15}{16}\frac{\left[\partial_a\left(\tilde{\rho}^2\right)\right]\mathcal{P}_{\rm III}^2|_{t=t_o}}{\tilde{\rho}_o^7} \right\}. \tag{98}$$

Generically, this can be written as

$$Q_a[\epsilon^0] = \sum_{n=1}^{3}\sum_{k=0}^{2n+1}\sum_{p=0}^{[k/2]} \frac{c_{nkp(a)}\Delta^{2n+1-k}\left(\phi-\phi_o\right)^{k-2p}\left(\theta-\frac{\pi}{2}\right)^{2p}}{\tilde{\rho}_o^{2n+1}}, \tag{99}$$

where $\Delta \equiv r - r_o$, and $c_{nkp(a)}$ is the coefficient of each individual term that depends on n, k and p as well as a, with a dimension $\mathcal{R}^{k-2}$ for $a = t,\, r$ and $\mathcal{R}^{k-1}$ for $a = \theta,\, \phi$.

The behavior of $Q_a[\epsilon^0]$, according to the powers of each factor on the right hand side of Eq. (99), is

$$Q_a[\epsilon^0] \sim \tilde{\rho}_o^{-(2n+1)}\Delta^{2n+1-k}\left(\phi-\phi_o\right)^{k-2p}\left(\theta-\frac{\pi}{2}\right)^{2p}\mathcal{R}^s, \tag{100}$$

where $s = k-2$ for $a = t,\, r$ and $s = k-1$ for $a = \theta,\, \phi$. Following the same procedure as in the beginning of Subsection V B, Eq. (100) becomes

$$Q_a[\epsilon^0] \sim \tilde{\rho}_o^{-(2n+1)}\Delta^{2n+1-2p-i}\left(\sin\Theta\right)^{2p+i}\left(\sin\Phi\right)^{2p}\left(\cos\Phi\right)^{i}\mathcal{R}^s, \tag{101}$$

where a binomial expansion over the index $i = 0,\, 1,\, \cdots,\, k-2p$ is assumed, and $s = 2p+i-2$ for $a = t,\, r$ and $s = 2p+i-1$ for $a = \theta,\, \phi$. Here we have disregarded any by-products like $O[(x-x_o)^{k-2p+2}]$ and $O[(x-x_o)^{2p+2}]$, which originate from $\left(\phi-\phi_o\right)^{k-2p}$ and $\left(\theta-\frac{\pi}{2}\right)^{2p}$, respectively when we rotate the angles: by putting them back into Eq. (100) we simply obtain ϵ^2-terms, which would correspond to $O(\ell^{-4})$ in Eq. (10) and should vanish when summed over ℓ in our final self-force calculation by Eq. (9). Then, the rest of the argument is developed in the same way as in the beginning of Subsection V B:

(i) $i = 2j+1$ ($j = 0,\, 1,\, 2,\, \cdots$)
The integrand for "$\langle\,\rangle$" process, $F(\Phi) \equiv \left(\cos\Phi\right)^{2j+1}\left(\sin\Phi\right)^{2p}$ has the property $F(\Phi+\pi) = -F(\Phi)$. Thus

$$\left\langle Q_a[\epsilon^0]\right\rangle = 0, \tag{102}$$

(ii) $i = 2j$ ($j = 0,\, 1,\, 2,\, \cdots$)
We have

$$Q_a[\epsilon^0] \sim \left(\sin\Phi\right)^{2p}\left(\cos\Phi\right)^{2j}\tilde{\rho}_o^{-2(n-q)-1}\Delta^{2(n-q)+1}\mathcal{R}^s, \tag{103}$$

where $q = 0,\, 1,\, \cdots,\, p+j$ is the index for a binomial expansion and $s = -2$ for $a = t,\, r$ and $s = -1$ for $a = \theta,\, \phi$. Here we can guarantee that $n - q \geq -\frac{1}{2}$, i.e. $n - q = 0,\, 1,\, 2,\, \cdots$ since $0 \leq q \leq p+j = p+\frac{1}{2}i$, $0 \leq i \leq k-2p$, and $p \leq k \leq 2n+1$. Then, Eq. (103) can be subcategorized into the following two cases;

(ii-1) $n-q\geq 1$
By Eqs. (39), (41), and (46)

$$Q_a[\epsilon^0] \underset{\Delta\to 0}{\sim} (\sin\Phi)^{2p}\,(\cos\Phi)^{2j}\,\Delta^2 P_\ell(\cos\Theta)\mathcal{R}^s \longrightarrow 0, \tag{104}$$

(ii-2) $n-q=0$
By Eqs. (39), (41), and (47)

$$Q_a[\epsilon^0] \underset{\Delta\to 0}{\sim} (\sin\Phi)^{2p}\,(\cos\Phi)^{2j}\,\Delta P_\ell(\cos\Theta)\mathcal{R}^s \longrightarrow 0, \tag{105}$$

where $s=-2$ for $a=t,\, r$ and $s=-1$ for $a=\theta,\,\phi$.

Clearly, in any cases the quantity $Q_a[\epsilon^0]$ does not survive, therefore we can conclude that C_a-terms are always zero. Q. E. D.

Also, this justifies the argument that we need not clarify the term $O[(x-x_{\rm o})^3]$ in Eq. (21) and its contribution to ρ^2, which is $O[(x-x_{\rm o})^4]$ in Eqs. (28) and (29) in Section IV or $\mathcal{P}_{\rm IV}$ in Eqs. (31) and (42) in Section V: by the analysis of the generic structure given above, $-\frac{1}{2}\,\partial_a\mathcal{P}_{\rm IV}|_{t=t_{\rm o}}/\tilde{\rho}_{\rm o}^3$ or $\frac{3}{4}\left[\partial_a\left(\tilde{\rho}^2\right)\right]\mathcal{P}_{\rm IV}\big|_{t=t_{\rm o}}/\tilde{\rho}_{\rm o}^5$ would simply vanish in the coincidence limit $x\to x_{\rm o}$, regardless of what $\mathcal{P}_{\rm IV}$ is.

VI. DISCUSSION

The scalar field in the flat spacetime limit

One interesting fact is that when we consider flat spacetime as the background our singular source field $\psi^{\rm S}$ may be determined in a completely different way from the case of curved spacetime. This flat spacetime version of $\psi^{\rm S}$, though obtained using a different method, should agree with its curved spacetime version when the flat spacetime limit is taken.

Without introducing the special coordinate frame like that of the THZ coordinates as employed in Section IV, this argument can be shown rather straightforward: the retarded field, which is equivalent to the singular source field in flat spacetime, can be computed directly using the coulomb potential and considering the special relativity. Here we will focus on $\rho=|\vec{x}-\vec{x}_{\rm o}|$ between the field point x and the source point $x_{\rm o}$. This will be expressed in terms of the Cartesian coordinates first. Then, we will switch from the Cartesian coordinates to the spherical polar coordinates and arrange the terms in a polynomial according to their orders, where quartic or higher order terms will be separated as errors. Finally, the expression of ρ^2 for $\psi_{\rm ret}$ in flat spacetime obtained thus will be shown to agree with the flat spacetime limit of ρ^2 for $\psi_{\rm S}$ in Eq. (29).

In the frame of reference in which the particle is always at rest at the point $\vec{x_{\rm o}}$, the retarded field at $\vec{x}$ is

$$\psi_{\rm ret} = \frac{q}{|\vec{x}-\vec{x}_{\rm o}|}.$$

We boost this frame such that in the new frame of reference the particle is moving with the 3-dim velocity $\vec{\beta}$ (see Ref. **(author?)** [14]):

$$\begin{aligned} T\equiv t-t_{\rm o} \quad &\Longrightarrow \quad T=\gamma\left(T'-\vec{\beta}\cdot\vec{X}'\right), \\ \vec{X}\equiv\vec{x}-\vec{x}_{\rm o} \quad &\Longrightarrow \quad \vec{X}=\vec{X}'+\vec{\beta}\left[\frac{\gamma-1}{\beta^2}\left(\vec{\beta}\cdot\vec{X}'\right)-\gamma T'\right], \end{aligned}$$

where $\vec{X}'\equiv\vec{x}'-\vec{x}'_{\rm o}$ and $T'\equiv t'-t'_{\rm o}$. Then, we have

$$\rho^2\equiv|\vec{x}-\vec{x}_{\rm o}|^2=\left\{(\vec{x}'-\vec{x}'_{\rm o})+\vec{\beta}\left[\frac{\gamma-1}{\beta^2}\left[\vec{\beta}\cdot(\vec{x}'-\vec{x}'_{\rm o})\right]-\gamma\,(t'-t'_{\rm o})\right]\right\}^2.$$

Dropping the $'$ notation and expanding the terms inside $\{\}$ out

$$\rho^2 = \gamma^2\beta^2 (t-t_o)^2 - 2\gamma^2 (t-t_o)\left[\vec{\beta}\cdot(\vec{x}-\vec{x}_o)\right] + \gamma^2\left[\vec{\beta}\cdot(\vec{x}-\vec{x}_o)\right]^2 + (\vec{x}-\vec{x}_o)^2 .$$

Now we need convert this expression into the spherical polar representation. First, using the cosine law, the distance between $\vec{x} = (r,\, \theta,\, \phi)$ and $\vec{x}_o = (r_o,\, \theta_o,\, \phi_o)$ can be expressed as

$$|\vec{x}-\vec{x}_o|^2 = r^2 + r_o^2 - 2rr_o\cos\vartheta,$$

where

$$\cos\vartheta = \sin\theta\sin\theta_o\cos(\phi-\phi_o) + \cos\theta\cos\theta_o,$$

which is obvious from the trigonometric rule. In particular, for the particle moving along an equatorial orbit ($\theta_o = \pi/2$) we may rewrite the above as

$$\begin{aligned}|\vec{x}-\vec{x}_o|^2 &= (r-r_o)^2 + r_o(r-r_o)\left(\theta-\frac{\pi}{2}\right)^2 + r_o(r-r_o)(\phi-\phi_o)^2 + r_o^2\left(\theta-\frac{\pi}{2}\right)^2 \\ &\quad + r_o^2(\phi-\phi_o)^2 + O[(\theta-\pi/2,\, \phi-\phi_o)^4],\end{aligned}$$

where all the trigonometric functions of the small arguments are expanded in Taylor series up to the cubic order.

To compute $\vec{\beta}\cdot(\vec{x}-\vec{x}_o)$, first of all one need change the basis vectors from $\{\hat{\mathbf{x}},\, \hat{\mathbf{y}},\, \hat{\mathbf{z}}\}$ to $\{\hat{\mathbf{r}}_o,\, \hat{\theta}_o,\, \hat{\phi}_o\}$ via

$$\begin{pmatrix}\hat{\mathbf{x}}\\ \hat{\mathbf{y}}\\ \hat{\mathbf{z}}\end{pmatrix} = \begin{pmatrix}\cos\phi_o & 0 & -\sin\phi_o\\ \sin\phi_o & 0 & \cos\phi_o\\ 0 & -1 & 0\end{pmatrix}\begin{pmatrix}\hat{\mathbf{r}}_o\\ \hat{\theta}_o\\ \hat{\phi}_o\end{pmatrix}.$$

Then, $\vec{x}-\vec{x}_o$ can be rewritten as

$$\begin{aligned}\vec{x}-\vec{x}_o &= (r\sin\theta\cos\phi)\left(\hat{\mathbf{r}}_o\cos\phi_o - \hat{\phi}_o\sin\phi_o\right) + (r\sin\theta\sin\phi)\left(\hat{\mathbf{r}}_o\sin\phi_o + \hat{\phi}_o\cos\phi_o\right) \\ &\quad - r\cos\theta\hat{\theta}_o - r_o\hat{\mathbf{r}}_o.\end{aligned}$$

Also, in the new basis the 3-dim velocity of the particle moving in the equatorial plane is expressed as

$$\vec{\beta} = \beta_r\hat{\mathbf{r}}_o + \beta_\phi\hat{\phi}_o.$$

Thus, taking a dot product of $\vec{\beta}$ and $\vec{x}-\vec{x}_o$ gives

$$\vec{\beta}\cdot(\vec{x}-\vec{x}_o) = \beta_r\left[r\sin\theta\cos(\phi-\phi_o) - r_o\right] + \beta_\phi r\sin\theta\sin(\phi-\phi_o).$$

Again, taking a series expansion of this quantity around the source point $\vec{x}_o = \left(r_o,\, \frac{\pi}{2},\, \phi_o\right)$, sufficiently up to the quadratic order, we have

$$\begin{aligned}\vec{\beta}\cdot(\vec{x}-\vec{x}_o) &= \beta_r(r-r_o) + \beta_\phi r_o(\phi-\phi_o) + \beta_\phi(r-r_o)(\phi-\phi_o) - \frac{1}{2}r_o\beta_r\left(\theta-\frac{\pi}{2}\right)^2 \\ &\quad - \frac{1}{2}r_o\beta_r(\phi-\phi_o)^2 + O[(\theta-\pi/2,\, \phi-\phi_o)^3].\end{aligned}$$

Finally, putting all the terms together we obtain the following

$$\begin{aligned}\rho^2 &= \left(E^2-1\right)(t-t_o)^2 - 2E\dot{r}(t-t_o)(r-r_o) - 2EJ(t-t_o)(\phi-\phi_o) \\ &\quad + \left(1+\dot{r}^2\right)(r-r_o)^2 + 2J\dot{r}(r-r_o)(\phi-\phi_o) + \left(r_o^2+J^2\right)(\phi-\phi_o)^2 + r_o^2\left(\theta-\frac{\pi}{2}\right)^2 \\ &\quad - \frac{2EJ}{r_o}(t-t_o)(r-r_o)(\phi-\phi_o) + r_oE\dot{r}(t-t_o)(\phi-\phi_o)^2 + r_oE\dot{r}(t-t_o)\left(\theta-\frac{\pi}{2}\right)^2 \\ &\quad + \frac{2J\dot{r}}{r_o}(r-r_o)^2(\phi-\phi_o) + r_o\left(1-\dot{r}^2+\frac{2J^2}{r_o^2}\right)(r-r_o)(\phi-\phi_o)^2 \\ &\quad + r_o\left(1-\dot{r}^2\right)(r-r_o)\left(\theta-\frac{\pi}{2}\right)^2 - r_oJ\dot{r}(\phi-\phi_o)^3 - r_oJ\dot{r}(\phi-\phi_o)\left(\theta-\frac{\pi}{2}\right)^2 \\ &\quad + O[(t-t_o,\, r-r_o,\, \theta-\pi/2,\, \phi-\phi_o)^4].\end{aligned}$$

This is exactly equal to the flat spacetime limit of ρ^2 for ψ^{S} when $M=0$ in Eq. (29), with appropriate replacement for the coefficients, using $E = (dt/d\tau)_o = \gamma$, $J = r_o^2\,(d\phi/d\tau)_o = \gamma r_o\beta_\phi$, and $\dot{r} = (dr/d\tau)_o = \gamma\beta_r$ together with the identity $\dot{r}^2 = E^2 - f\left(1+J^2/r_o^2\right)$.

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Appendix A: THE THZ NORMAL COORDINATES

In Section IV we have introduced the Thorne-Hartle-Zhang's normal coordinates to simplify the description of $\psi^{\rm S}$ as if it was measured by an observer who travels on a particle moving in curved spacetime: in this coordinate system $\{\mathcal{X}^A\}(A=0,\,1,\,2,\,3)$, $\psi^{\rm S}=q/\rho+O(\rho^2/\mathcal{R}^3)$, where $\rho^2\equiv\delta_{IJ}\mathcal{X}^I\mathcal{X}^J$ $(I,\,J=1,\,2,\,3)$. As presented by Eq. (29), the specification of $O[(x-x_{\rm o})^4]$ in ρ^2 (thus of $O[(x-x_{\rm o})^3]$ in $\mathcal{X}^I$ in Eq. (21)) is not necessary for our current mode-sum regularization scheme: it proves not to contribute to the regularization parameters A_a, B_a, and C_a.

In the practical calculation of self-force, however, it will be very useful to extend our scheme to the next orders, i.e. to D_a or higher terms. If we extend Eq. (43), say, to ϵ^1-term, it would generate the next-order regularization terms $-2\sqrt{2}D_a/[(2\ell-1)(2\ell+3)]$ in the place of $O(\ell^{-2})$ in Eq. (10). Strictly, these terms would give non-vanishing contributions to the self-force since the sum of $-2\sqrt{2}D_a/[(2\ell-1)(2\ell+3)]$ is taken over many but finite number of ℓ's in actual numerical calculations.

In order to determine D_a or higher terms, the knowledge of $O[(x-x_{\rm o})^5]$ in ρ^2 and thus of $O[(x-x_{\rm o})^4]$ in $\mathcal{X}^I$ will be required. As it should be an important tool for this purpose, we present below a more detailed description of the THZ normal coordinates $\{\mathcal{X}^A\}(A=0,\,1,\,2,\,3)$ for a particle moving along a general orbit (confined to the equatorial plane for convenience) about a Schwarzschild black hole [22]: the expressions are given in terms of the Schwarzschild coordinates $x^a=(t,\,r,\,\theta,\,\phi)$ and specified up to the quortic order

$$\begin{cases} \mathcal{T}\equiv\mathcal{X}^0=-u_{{\rm o}A}\left[X^A+\alpha^A{}_{BCD}X^BX^CX^D+\beta^A{}_{BCDE}X^BX^CX^DX^E\right]+O(X^5), \\ \mathcal{X}^I=n^{(I)}_{{\rm o}A}\left[X^A+\kappa^A{}_{BCD}X^BX^CX^D+\lambda^A{}_{BCDE}X^BX^CX^DX^E\right]+O(X^5) \end{cases} \tag{A1}$$

with

$$X^A=M^A{}_a(x^a-x^a_{\rm o})+\frac{1}{2}M^A{}_a\,\Gamma^a_{bc}\big|_{\rm o}(x^b-x^b_{\rm o})(x^c-x^c_{\rm o})+O[(x-x_{\rm o})^3], \tag{A2}$$

$$\begin{aligned}\alpha^A{}_{BCD}\ \equiv\ &\frac{1}{6}\,\Gamma^A_{PQ,R}\big|_{\rm o}\left(\pi^P{}_B\pi^Q{}_C\pi^R{}_D+3\pi^Q{}_B\pi^R{}_Ch^P{}_D\right.\\ &\left.+3\pi^R{}_Bh^P{}_Ch^Q{}_D+h^P{}_Bh^Q{}_Ch^R{}_D\right),\end{aligned} \tag{A3}$$

$$\begin{aligned}\beta^A{}_{BCDE}\ \equiv\ &\frac{1}{24}\,\Gamma^A_{PQ,RS}\big|_{\rm o}\left(\pi^P{}_B\pi^Q{}_C\pi^R{}_D\pi^S{}_E+4\pi^Q{}_B\pi^R{}_C\pi^S{}_Dh^P{}_E\right.\\ &\left.6\pi^R{}_B\pi^S{}_Ch^P{}_Dh^Q{}_E+4\pi^S{}_Bh^P{}_Ch^Q{}_Dh^R{}_E+h^P{}_Bh^Q{}_Ch^R{}_Dh^S{}_E\right)\\ &-\frac{5}{168}\,R^A{}_{PQR,S}\big|_{\rm o}\,\pi^{QS}h^P{}_Bh^R{}_Ch_{DE},\end{aligned} \tag{A4}$$

$$\begin{aligned}\kappa^A{}_{BCD}\ \equiv\ &\frac{1}{6}\,\Gamma^A_{PQ,R}\big|_{\rm o}\left(\pi^P{}_B\pi^Q{}_C\pi^R{}_D+3\pi^Q{}_B\pi^R{}_Ch^P{}_D\right.\\ &\left.+3\pi^R{}_Bh^P{}_Ch^Q{}_D+h^P{}_Bh^Q{}_Ch^R{}_D\right)\\ &-R^E{}_{PQR}\big|_{\rm o}\left(\frac{1}{6}\delta^A{}_E\pi^{PQ}h^R{}_Bh_{CD}+\frac{1}{3}\delta^A{}_B\pi^Q{}_Eh^P{}_Ch^R{}_D\right),\end{aligned} \tag{A5}$$

$$\begin{aligned}\lambda^A{}_{BCDE}\ \equiv\ &\frac{1}{24}\,\Gamma^A_{PQ,RS}\big|_{\rm o}\left(\pi^P{}_B\pi^Q{}_C\pi^R{}_D\pi^S{}_E+4\pi^Q{}_B\pi^R{}_C\pi^S{}_Dh^P{}_E\right.\\ &\left.6\pi^R{}_B\pi^S{}_Ch^P{}_Dh^Q{}_E+4\pi^S{}_Bh^P{}_Ch^Q{}_Dh^R{}_E+h^P{}_Bh^Q{}_Ch^R{}_Dh^S{}_E\right)\\ &-R^F{}_{PQR,S}\big|_{\rm o}\left(\frac{1}{6}\delta^A{}_F\pi^{PQ}\pi^S{}_Bh^R{}_Ch_{DE}+\frac{1}{3}\delta^A{}_B\pi^Q{}_F\pi^S{}_Ch^P{}_Dh^R{}_E\right.\\ &\left.+\frac{1}{24}\delta^A{}_F\pi^{PQ}h^R{}_Bh^S{}_Ch_{DE}+\frac{1}{24}\delta^A{}_B\pi^Q{}_Fh^P{}_Ch^R{}_Dh^S{}_E+\frac{2}{63}\delta^A{}_F\pi^{RS}h^P{}_Bh^Q{}_Ch_{DE}\right),\end{aligned} \tag{A6}$$

where $A, \cdots, E$ and $P, \cdots, S = 0, 1, 2, 3$, and $I, J, K, L = 1, 2, 3$. For Eq. (A1) we have

$$
\begin{aligned}
u_{\rm o}^{A} &= \left(f^{-1/2}E,\, f^{-1/2}\dot{r},\, \frac{J}{r_{\rm o}},\, 0\right), && \text{(A7)}\\
n_{\rm o}^{(1)A} &= \left(-f^{-1/2}\dot{r},\, 1+\frac{\dot{r}^2}{f^{1/2}E+f},\, \frac{J\dot{r}}{r_{\rm o}\left(E+f^{1/2}\right)},\, 0\right), && \text{(A8)}\\
n_{\rm o}^{(2)A} &= \left(-\frac{J}{r_{\rm o}},\, \frac{J\dot{r}}{r_{\rm o}\left(E+f^{1/2}\right)},\, 1+\frac{J^2}{r_{\rm o}^2\left(f^{-1/2}E+1\right)},\, 0\right), && \text{(A9)}\\
n_{\rm o}^{(3)A} &= (0,\, 0,\, 0,\, 1), && \text{(A10)}
\end{aligned}
$$

where $f = \left(1-\frac{2M}{r_{\rm o}}\right)$, and $E \equiv -u_t = (1-2M/r_{\rm o})\,(dt/d\tau)_{\rm o}$ (τ: proper time) and $J \equiv u_\phi = r_{\rm o}^2\,(d\phi/d\tau)_{\rm o}$ are the conserved energy and angular momentum in the background, respectively, and $\dot{r} \equiv u^r = (dr/d\tau)_{\rm o}$. In Eq. (A2) we define

$$
x_{\rm o}^{a} = \left(t_{\rm o},\, r_{\rm o},\, \frac{\pi}{2},\, \phi_{\rm o}\right), \tag{A11}
$$

$$
M^{A}{}_{a} = \mathrm{diag}\left[f^{1/2},\, f^{-1/2},\, r_{\rm o},\, -r_{\rm o}\right], \tag{A12}
$$

along with the non-zero Christoffel symbols at $x_{\rm o}$ in the Schwarzschild background

$$
\Gamma^{t}_{tr}\big|_{\rm o} = \frac{M}{fr_{\rm o}^2},\; \Gamma^{r}_{tt}\big|_{\rm o} = \frac{fM}{r_{\rm o}^2},\; \Gamma^{r}_{rr}\big|_{\rm o} = -\frac{M}{fr_{\rm o}^2},\; \Gamma^{r}_{\theta\theta}\big|_{\rm o} = -fr_{\rm o},\; \Gamma^{r}_{\phi\phi}\big|_{\rm o} = -fr_{\rm o},\; \Gamma^{\theta}_{r\theta}\big|_{\rm o} = \Gamma^{\phi}_{r\phi}\Big|_{\rm o} = \frac{1}{r_{\rm o}}. \tag{A13}
$$

The quantities $\Gamma^{A}_{BC,D}\big|_{\rm o}$, $R^{A}{}_{BCD}\big|_{\rm o}$, $\Gamma^{A}_{BC,DE}\big|_{\rm o}$ and $R^{A}{}_{BCD,E}\big|_{\rm o}$ in Eqs. (A3)-(A6) are evaluated from the initial static normal coordinates $\{X^A\}$ represented by Eq. (A2). They follow the identities

$$
\begin{aligned}
\Gamma^{A}_{BC,D}\big|_{\rm o} &= H^{A}{}_{BCD} + H^{A}{}_{CBD} - H_{BC}{}^{A}{}_{D}, && \text{(A14)}\\
R^{A}{}_{BCD}\big|_{\rm o} &= H_{BC}{}^{A}{}_{D} - H^{A}{}_{CBD} - H_{BD}{}^{A}{}_{C} + H^{A}{}_{DBC}, && \text{(A15)}\\
\Gamma^{A}_{BC,DE}\big|_{\rm o} &= 3\left(H^{A}{}_{BCDE} + H^{A}{}_{CBDE} - H_{BC}{}^{A}{}_{DE}\right), && \text{(A16)}\\
R^{A}{}_{BCD,E}\big|_{\rm o} &= 3\left(H_{BC}{}^{A}{}_{DE} - H^{A}{}_{CBDE} - H_{BD}{}^{A}{}_{CE} + H^{A}{}_{DBCE}\right), && \text{(A17)}
\end{aligned}
$$

where the building blocks H_{ABCD} and H_{ABCDE} are taken from $g_{AB} = \eta_{AB} + H_{ABCD}X^CX^D + H_{ABCDE}X^CX^DX^E$ (the linearized gravity in the geometry of $\{X^A\}$) and have symmetric properties $H_{ABCD} = H_{(AB)(CD)}$ and $H_{ABCDE} = H_{(AB)(CDE)}$. The non-zero H_{ABCD} and H_{ABCDE} turn out to be

$$
H_{0000} = -\frac{M^2}{fr_{\rm o}^4},\; H_{0011} = \frac{1}{f}\left(\frac{2M}{r_{\rm o}^3} - \frac{3M^2}{r_{\rm o}^4}\right),\; H_{0101} = \frac{M^2}{fr_{\rm o}^4},\; H_{0202} = H_{0303} = \frac{M}{2r_{\rm o}^3},
$$

$$
H_{1100} = \frac{M^2}{fr_{\rm o}^4},\; H_{1111} = \frac{1}{f}\left(\frac{2M}{r_{\rm o}^3} - \frac{M^2}{r_{\rm o}^4}\right),\; H_{1122} = H_{1133} = -\frac{f}{r_{\rm o}^2},\; H_{1212} = H_{1313} = -\frac{1}{2}\left(\frac{1}{r_{\rm o}^2} - \frac{M}{r_{\rm o}^3}\right),
$$

$$
H_{2222} = -\frac{f}{r_{\rm o}^2},\; H_{2233} = -\frac{1}{r_{\rm o}^2},\; H_{2323} = -\frac{f}{2r_{\rm o}^2},\; H_{3333} = -\frac{f}{r_{\rm o}^2},
$$

and

$$
H_{00001} = -\frac{1}{3f^{3/2}}\left(\frac{2M^2}{r_{\rm o}^5} - \frac{3M^3}{r_{\rm o}^6}\right),\; H_{00111} = -\frac{1}{f^{3/2}}\left(\frac{2M}{r_{\rm o}^4} - \frac{6M^2}{r_{\rm o}^5} + \frac{5M^3}{r_{\rm o}^6}\right),
$$

$$
H_{00122} = H_{00133} = \frac{1}{3f^{1/2}}\left(\frac{2M}{r_{\rm o}^4} - \frac{3M^2}{r_{\rm o}^5}\right),\; H_{01000} = -\frac{M^3}{f^{3/2}r_{\rm o}^6},\; H_{01011} = -\frac{1}{3f^{3/2}}\left(\frac{4M^2}{r_{\rm o}^5} - \frac{3M^3}{r_{\rm o}^6}\right),
$$

$$H_{01022} = H_{01033} = \frac{M^2}{3f^{1/2} r_o^5},\ H_{02012} = H_{03013} = -\frac{1}{6f^{1/2}}\left(\frac{M}{r_o^4} - \frac{3M^2}{r_o^5}\right),$$

$$H_{11001} = -\frac{1}{3f^{3/2}}\left(\frac{2M^2}{r_o^5} + \frac{3M^3}{r_o^6}\right),\ H_{11111} = -\frac{1}{f^{3/2}}\left(\frac{2M}{r_o^4} - \frac{6M^2}{r_o^5} + \frac{3M^3}{r_o^6}\right),$$

$$H_{11122} = H_{11133} = \frac{1}{3f^{1/2}}\left(\frac{4}{r_o^3} - \frac{14M}{r_o^4} + \frac{15M^2}{r_o^5}\right),\ H_{12002} = \frac{1}{6f^{1/2}}\left(\frac{M}{r_o^4} - \frac{M^2}{r_o^5}\right),$$

$$H_{12112} = \frac{1}{6f^{1/2}}\left(\frac{4}{r_o^3} - \frac{11M}{r_o^4} + \frac{9M^2}{r_o^5}\right),\ H_{12222} = -\frac{f^{1/2}}{2}\left(\frac{1}{r_o^3} - \frac{M}{r_o^4}\right),\ H_{12233} = \frac{f^{1/2}}{6}\left(\frac{1}{r_o^3} + \frac{M}{r_o^4}\right),$$

$$H_{13003} = \frac{1}{6f^{1/2}}\left(\frac{M}{r_o^4} - \frac{M^2}{r_o^5}\right),\ H_{13113} = \frac{1}{6f^{1/2}}\left(\frac{4}{r_o^3} - \frac{11M}{r_o^4} + \frac{9M^2}{r_o^5}\right),$$

$$H_{13223} = -\frac{f^{1/2}}{6}\left(\frac{1}{r_o^3} - \frac{M}{r_o^4}\right),\ H_{13333} = -\frac{f^{1/2}}{2}\left(\frac{1}{r_o^3} - \frac{M}{r_o^4}\right),$$

$$H_{22122} = H_{33133} = \frac{2f^{1/2}}{3}\left(\frac{1}{r_o^3} - \frac{3M}{r_o^4}\right),\ H_{22133} = \frac{2f^{1/2}}{3r_o^3},\ H_{23123} = \frac{f^{1/2}}{3}\left(\frac{1}{r_o^3} - \frac{3M}{r_o^4}\right).$$

According to Ref. (**(author?)** [13]), one can show the following out of the two geometries, the THZ $\{\mathcal{X}^A\}$ and the Schwarzschild $\{x^a\}$

$$\begin{aligned} \tilde{g}^{AB} &= g^{ab}\frac{\partial \mathcal{X}^A}{\partial x^a}\frac{\partial \mathcal{X}^B}{\partial x^b} \\ &= \eta^{AB} + O[(x - x_o)^2] \\ &= \eta^{AB} + O(\mathcal{X}^2), \end{aligned} \tag{A18}$$

where tilde denotes the THZ geometry and $O[(x-x_o)^2]$ is converted to $O(\mathcal{X}^2)$ via the inverse transformation of Eq. (A2). The metric perturbations $O(\mathcal{X}^2)$ in the last line of Eq. (A18) should take specific forms to satisfy the properties of the THZ coordinates as mentioned in Section IV. Our results show that

$$\tilde{g}_{00} = -1 - \mathcal{E}_{KL}\mathcal{X}^K\mathcal{X}^L - \frac{1}{3}\mathcal{E}_{KLM}\mathcal{X}^K\mathcal{X}^L\mathcal{X}^M + O(\rho^4/\mathcal{R}^4), \tag{A19}$$

$$\begin{aligned} \tilde{g}_{0I} &= \frac{2}{3}\epsilon_{IKP}\mathcal{B}^P{}_L\mathcal{X}^K\mathcal{X}^L - \frac{10}{21}\dot{\mathcal{E}}_{KL}\mathcal{X}^K\mathcal{X}^L\mathcal{X}_I + \frac{4}{21}\rho^2\dot{\mathcal{E}}_{KI}\mathcal{X}^K \\ &\quad + \frac{1}{3}\epsilon_{IKP}\mathcal{B}^P{}_{LM}\mathcal{X}^K\mathcal{X}^L\mathcal{X}^M + O(\rho^4/\mathcal{R}^4), \end{aligned} \tag{A20}$$

$$\begin{aligned} \tilde{g}_{IJ} &= \delta_{IJ} - \delta_{IJ}\mathcal{E}_{KL}\mathcal{X}^K\mathcal{X}^L + \frac{5}{21}\epsilon_{IKP}\dot{\mathcal{B}}^P{}_L\mathcal{X}^K\mathcal{X}^L\mathcal{X}_J \\ &\quad - \frac{1}{21}\rho^2\epsilon_{KPI}\dot{\mathcal{B}}_J{}^P\mathcal{X}^K - \frac{1}{3}\delta_{IJ}\mathcal{E}_{KLM}\mathcal{X}^K\mathcal{X}^L\mathcal{X}^M + O(\rho^4/\mathcal{R}^4), \end{aligned} \tag{A21}$$

where $\rho^2 = \mathcal{X}^2 + \mathcal{Y}^2 + \mathcal{Z}^2$ and indices $I,\ J,\ K,\ L,\ M,\ P = 1, 2, 3$. The external multipole moments are spatial, symmetric, tracefree tensors and are related to the Riemann tensor evaluated on the particle's worldline by

$$\mathcal{E}_{IJ} = \tilde{R}_{0I0J}\Big|_o, \tag{A22}$$

$$\mathcal{B}_{IJ} = \frac{1}{2}\epsilon_I{}^{PQ}\,\tilde{R}_{PQJ0}\Big|_o, \tag{A23}$$

$$\mathcal{E}_{IJK} = \left[\nabla_K\,\tilde{R}_{0I0J}\Big|_o\right]^{\rm STF}, \tag{A24}$$

$$\mathcal{B}_{IJK} = \frac{3}{8}\left[\epsilon_I{}^{PQ}\nabla_K\,\tilde{R}_{PQJ0}\Big|_o\right]^{\rm STF}. \tag{A25}$$

where STF means to take the symmetric and tracefree part with respect to the spatial indices I, J, K. The dot denotes differentiation of the multipole moment with respect to $\mathcal{T}$ along the particle's worldline. One should see that $\mathcal{E}_{IJ} \sim \mathcal{B}_{IJ} \sim O(1/\mathcal{R}^2)$ and $\mathcal{E}_{IJK} \sim \mathcal{B}_{IJK} \sim \dot{\mathcal{E}}_{IJ} \sim \dot{\mathcal{B}}_{IJ} \sim O(1/\mathcal{R}^3)$ for consistency of the dimensions. The fact that all of the above external multipole moments are tracefree follows from the assumption that the background geometry is a vacuum solution of the Einstein equations. These results agree with Eqs. (17) and (18) of Paper I **(author?)** [7] or Eqs. (3.26a)-(3.26c) of Zhang **(author?)** [9] to the lowest a few orders.

Appendix B: HYPERGEOMETRIC FUNCTIONS AND REPRESENTATIONS OF REGULARIZATION PARAMETERS

In Section V we define

$$\chi \equiv 1 - \alpha \sin^2 \Phi \tag{B1}$$

with

$$\alpha \equiv \frac{J^2}{r_{\rm o}^2 + J^2}. \tag{B2}$$

And we use

$$\begin{aligned} \left\langle \chi^{-p} \right\rangle \equiv \left\langle \left(1 - \alpha \sin^2 \Phi\right)^{-p} \right\rangle &= \frac{2}{\pi} \int_0^{\pi/2} \left(1 - \alpha \sin^2 \Phi\right)^{-p} d\Phi \\ &= {}_2F_1\left(p, \frac{1}{2}; 1, \alpha\right) \equiv F_p. \end{aligned} \tag{B3}$$

In particular, for the cases $p = \frac{1}{2}$ and $p = -\frac{1}{2}$ we have the following representations

$$F_{1/2} = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}, 1; \alpha\right) = \frac{2}{\pi} \hat{K}(\alpha) \tag{B4}$$

and

$$F_{-1/2} = {}_2F_1\left(-\frac{1}{2}, \frac{1}{2}, 1; \alpha\right) = \frac{2}{\pi} \hat{E}(\alpha), \tag{B5}$$

where $\hat{K}(\alpha)$ and $\hat{E}(\alpha)$ are called complete elliptic integrals of the first and second kinds, respectively.

If we take the derivative of $F_{1/2}$ with respect to $k \equiv \sqrt{\alpha}$ via Eq. (B3), we obtain

$$\frac{\partial F_{1/2}}{\partial k} = -\frac{F_{1/2}}{k} + \frac{F_{3/2}}{k}, \tag{B6}$$

or using Eq. (B4)

$$\frac{\partial \hat{K}}{\partial k} = -\frac{\hat{K}}{k} + \frac{\pi}{2} \frac{F_{3/2}}{k}. \tag{B7}$$

However, Ref. **(author?)** [16] shows that

$$\frac{\partial \hat{K}}{\partial k} = \frac{\hat{E}}{k\left(1 - k^2\right)} - \frac{\hat{K}}{k}. \tag{B8}$$

Thus, by comparing Eq. (B7) and Eq. (B8) we find the representation

$$F_{3/2} = \frac{2}{\pi} \frac{\hat{E}}{1 - k^2} = \frac{2}{\pi} \frac{\hat{E}}{1 - \alpha}. \tag{B9}$$

Further, we can also find the representation for $F_{5/2}$. First, taking the derivative of $F_{3/2}$ with respect to $k \equiv \sqrt{\alpha}$ via Eq. (B3) gives

$$\frac{\partial F_{3/2}}{\partial k} = -\frac{3F_{3/2}}{k} + \frac{3F_{5/2}}{k}. \tag{B10}$$

Also, using Eq. (B9) together with Eqs. (B3)-(B5), another expression for the same derivative is obtained solely in terms of complete elliptic integrals

$$\frac{\partial F_{3/2}}{\partial k} = \frac{2}{\pi} \frac{\left(1+k^2\right)\hat{E} - \left(1-k^2\right)\hat{K}}{k\left(1-k^2\right)^2}. \tag{B11}$$

Then, by Eqs. (B9), (B10), and, (B11) we find

$$F_{5/2} = \frac{2}{3\pi}\left[\frac{2\left(2-\alpha\right)\hat{E}}{\left(1-\alpha\right)^2} - \frac{\hat{K}}{1-\alpha}\right]. \tag{B12}$$

Now, using Eqs. (B4), (B9), and (B12), we may rewrite the non-zero B-terms of regularization parameters, Eqs. (14)-(16) in Section III as

$$B_t = \frac{q^2}{r_\mathrm{o}^2} \frac{E\dot{r}\left[\hat{K}(\alpha) - 2\hat{E}(\alpha)\right]}{\pi\left(1+J^2/r_\mathrm{o}^2\right)^{3/2}}, \tag{B13}$$

$$B_r = \frac{q^2}{r_\mathrm{o}^2} \frac{\left(\dot{r}^2 - 2E^2\right)\hat{K}(\alpha) + \left(\dot{r}^2 + E^2\right)\hat{E}(\alpha)}{\pi\left(1-2M/r_\mathrm{o}\right)\left(1+J^2/r_\mathrm{o}^2\right)}, \tag{B14}$$

$$B_\phi = \frac{q^2}{r_\mathrm{o}} \frac{\dot{r}\left[\hat{K}(\alpha) - \hat{E}(\alpha)\right]}{\left(J/r_\mathrm{o}\right)\left(1+J^2/r_\mathrm{o}^2\right)^{1/2}}, \tag{B15}$$

which are exactly the same to the results of Barack and Ori **(author?)** [10].

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[17] D.-H. Kim and S. Detweiler, "Regularization parameters for gravitational self-force calculations in the Schwarzschild geometry" (in preparation)
[18] Or alternatively, one can use the arguement $\frac{\partial}{\partial\Theta}\big|_\Delta P_\ell(\cos\Theta) = 0$ as $\Theta \to 0$, to show that this part does not survive at the end.
[19] The by-product from $O[(x-x_\mathrm{o})^4]$ in Eq. (53), that is to say, $-\frac{q^2}{2}\tilde{\rho}_\mathrm{o}{}^{-3}O[(x-x_\mathrm{o})^3]$ is suppressed since this can be categorized into the C_a-term group (see Subsection V C).

[20] The by-product from $O[(x-x_{\rm o})^3]$ in Eq. (57), that is to say, $-\frac{q^2}{2}\tilde{\rho}_{\rm o}^{-3}O[(x-x_{\rm o})^3]$ is suppressed since this can be categorized into the C_a-term group (see Subsection V C).

[21] Developing our mode-sum regularization scheme further, we may include ϵ^1-term in Eq. (43) and this would generate the next-order regularization terms $-2\sqrt{2}D_a/[(2\ell-1)(2\ell+3)]$ in the place of $O(\ell^{-2})$ in Eq. (10). Rigorously, their contributions to the self-force would be non-vanishing since we are taking the sum of $-2\sqrt{2}D_a/[(2\ell-1)(2\ell+3)]$ over many but finite number of ℓ's. Thorough discussions on D_a-terms for a circular and for a general orbit cases are found in Ref. **(author?)** [7] and Ref. **(author?)** [17], respectively.

[22] For derivations of the THZ-coordinates and for discussions on their application to D_a-terms, readers may be referred to Ref. **(author?)** [7] and Ref. **(author?)** [17] for a circular and for a general orbit cases, respectively.